

# Towards a derivation of holographic entanglement entropy

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**ABSTRACT:** We provide a derivation of holographic entanglement entropy for spherical entangling surfaces. Our construction relies on conformally mapping the boundary CFT to a hyperbolic geometry and observing that the vacuum state is mapped to a thermal state in the latter geometry. Hence the conformal transformation maps the entanglement entropy to the thermodynamic entropy of this thermal state. The AdS/CFT dictionary allows us to calculate this thermodynamic entropy as the horizon entropy of a certain topological black hole. In even dimensions, we also demonstrate that the universal contribution to the entanglement entropy is given by  $A$ -type trace anomaly for any CFT, without reference to holography.

**KEYWORDS:** [entanglement entropy](#), [conformal anomaly](#), [holography](#).

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## 1. Introduction

Entanglement entropy has become an important quantity for the study of quantum matter. It allows one to distinguish new topological phases and characterize critical points, *e.g.*, [1, 2, 3]. Entanglement entropy has also been considered in discussions of holographic descriptions of quantum gravity, in particular, for the AdS/CFT correspondence [4, 5]. In this context, as well as characterizing new properties of holographic field theories, *e.g.*, [6], it has been suggested that entanglement entropy may provide new insights into the quantum structure of spacetime [7].

The proposal [4, 5] of how to calculate holographic entanglement entropy is both simple and elegant. In the boundary field theory, one would begin by choosing a particular spatial region  $V$ . The entanglement entropy between  $V$  and its complement  $\bar{V}$  would then be the von Neumann entropy of the density matrix which results upon integrating out the field theory degrees of freedom in  $\bar{V}$ . In the holographic calculation

which is proposed to yield the same entropy, one considers bulk surfaces  $m$  which are "homologous" [8, 9] to the region  $V$  in the boundary (in particular  $\partial m = \partial V$ ). Then one extremizes the area<sup>1</sup> of  $m$  to calculate the entanglement entropy:

$$S(V) = \text{ext}_{m \sim V} \left[ \frac{A(m)}{4G_N} \right]. \quad (1.1)$$

An implicit assumption in eq. (1.1) is that the bulk physics is described by (classical) Einstein gravity. Hence we might note the similarity between this expression (1.1) and that for black hole entropy. While this proposal to calculate the holographic entanglement entropy passes a variety of consistency tests, *e.g.*, see [8, 10, 11], there is no concrete construction which allows one to derive this holographic formula (1.1).

A standard approach to calculating entanglement entropy in field theory makes use of the replica trick [2, 12]. This approach begins by calculating the partition function on an  $n$ -fold cover of the background geometry where a cut is introduced throughout the exterior region  $\bar{V}$ . If one were to apply this construction for the boundary CFT in holographic framework, one would naturally produce a conical singularity in the bulk geometry with an angular excess of  $2\pi(n-1)$ . However, without a full understanding of string theory or quantum gravity in the AdS bulk, we do not understand how to resolve the resulting conical curvature singularity and so it is not really possible work with this bulk geometry in a controlled way, *e.g.*, it is not possible to properly evaluate the saddle-point action in the gravity theory. This issue was emphasized in [8] in critiquing the attempted derivation of [9]. Further, it was demonstrated there that the approach of [9] leads to incorrect results in holographic calculations of Renyi entropies.

Hence the replica trick does not seem a useful starting point in considering a derivation of holographic entanglement entropy. An interesting derivation of the holographic entanglement entropy for a specific geometry was recently presented in [13, 14]. In particular, the boundary CFT was placed on an  $R \times S^{d-1}$  background and the entanglement entropy was calculated for an entangling surface that divided the sphere into two halves. It was argued that the holographic entanglement entropy was given by the horizon entropy of a certain topological black hole whose horizon divided the bulk geometry in half. In this paper, we clarify this construction and in doing so extend the derivation to more general spherical entangling surfaces.

In section 2, we set aside holography and begin with a discussion of the entanglement entropy for spherical entangling surfaces for a general conformal field theory

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<sup>1</sup>If the calculation is done in a Minkowski signature background, the extremal area is only a saddle point. However, if one first Wick rotates to Euclidean signature, the extremal surface will yield the minimal area. In either case, the area must be suitably regulated to produce a finite answer. Further note that for a  $d$ -dimensional boundary theory, the bulk has  $d+1$  dimensions while the surface  $m$  has  $d-1$  dimensions. We are using 'area' to denote the  $(d-1)$ -dimensional volume of  $m$ .

(CFT). In particular, we use conformal transformations to map the causal development of the interior region  $V$  to a new geometry which is the direct product of time with the hyperbolic plane  $H^{d-1}$ . Further, we demonstrate that if the CFT began in the vacuum in the original space, in this new geometry, we have a thermal bath whose temperature is controlled by the size of the sphere. Hence the density matrix describing the CFT inside the sphere becomes a thermal density matrix on the hyperbolic geometry (up to a unitary transformation which, of course, preserves the entropy). Hence the entanglement entropy across the sphere becomes the thermodynamic entropy in the second space. This construction was, in part, inspired by the appearance of a hyperbolic space in the calculations of entanglement entropies across spheres for a free scalar in [15] and it generalizes some older observations in ref. [16], again for free field theories.

While the results outlined above are quite general, in particular applying for any arbitrary CFT, it may seem this discussion only replaces one difficult problem, the calculation of the entanglement entropy, with another equally difficult problem, the calculation of the thermal entropy in a hyperbolic space. However, we are particularly interested in applying this result to the AdS/CFT correspondence. In this framework, the standard holographic dictionary suggests that the thermal state in the boundary CFT is dual to a black hole in the bulk gravity theory. Hence in section 3, we identify the corresponding black hole dual to the thermal bath on the hyperbolic geometry. The latter turns out to be a certain topological black hole with a hyperbolic horizon, but also one which is simply a hyperbolic foliation of the empty AdS spacetime. Hence with this bulk interpretation, we are able to compute the entanglement entropy as the horizon entropy of the black hole. This construction is not limited to having Einstein gravity in the bulk and so in fact, a general result is presented for any gravitational theory.

A generalization of the replica trick [12] has been applied to relate entanglement entropy to the trace anomaly for CFT's in an even number of spacetime dimensions [5, 17]. To be precise, with a general entangling surface, the universal coefficient of the logarithmic contribution to the entanglement entropy is given by some linear combination of the central charges appearing in the trace anomaly for any CFT with even  $d$ . The latter can be written as [18]

$$\langle T^\mu{}_\mu \rangle = \sum B_n I_n - 2(-)^{d/2} A E_d. \quad (1.2)$$

where  $E_d$  is the Euler density in  $d$  dimensions and  $I_n$  are the independent Weyl invariants of weight  $-d$ .<sup>2</sup> A result of our holographic analysis in section 3 is that the

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<sup>2</sup>Note that we have discarded a scheme dependent total derivative on the right-hand side of eq. (1.2). For more details on our conventions here, see [14].

linear combination determining the universal contribution for a spherical entangling surface reduces to be simply the coefficient of the  $A$ -type trace anomaly. In section 4, we obtain the same result for an arbitrary CFT, without reference to holography. There, our approach again relies on using a conformal mapping of the geometry in the entanglement entropy calculation. However, in this case, our conformal mapping takes the CFT to (the static patch of) de Sitter space, where the state is again thermal, and the entropy is again interpreted as ordinary thermodynamical entropy.

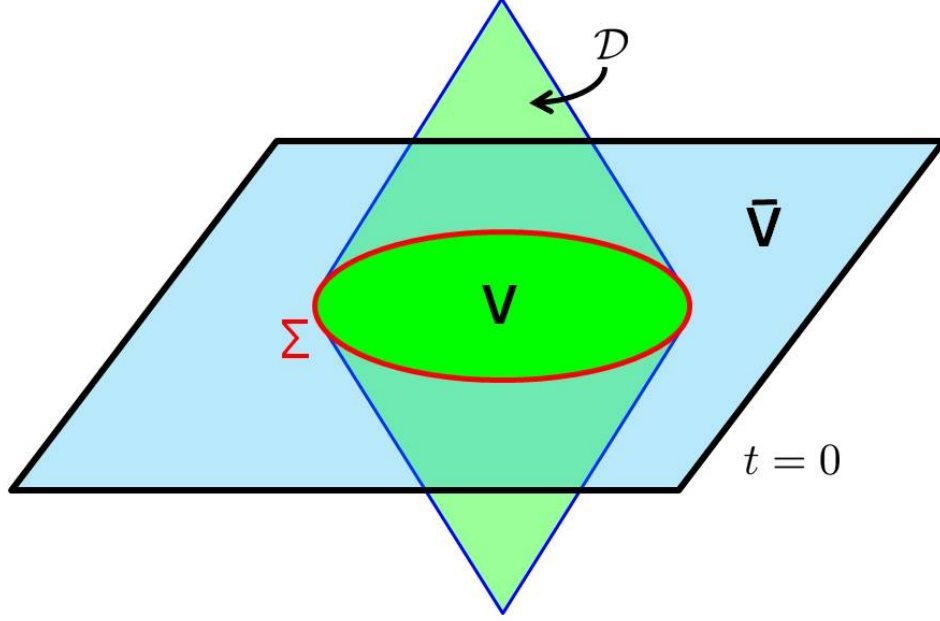
A connection of the coefficient of the logarithmic term in the entanglement entropy of a sphere with the  $A$ -type anomaly was also found previously for  $d = 4$  by [17]. Further, this connection was also noted for free scalar fields in any even dimension in [19]. Let us note that there have been a number of other recent works related to the entanglement entropy of free fields for a spherical entangling surface [15, 20, 21, 22].

## 2. The CFT story

In this section, we describe the entanglement entropy of any general conformal field theory for spherical entangling surfaces in certain background geometries. We emphasize that the discussion here is made purely in the context of quantum field theory (QFT), without reference to holography. However, the results found here will set the stage of a holographic calculation of the same entanglement entropy using the AdS/CFT correspondence in section 3. In particular, we consider here a  $d$ -dimensional CFT in Minkowski space  $R^{1,d-1}$  and examine the entanglement entropy across a spherical surface  $S^{d-2}$ . As described above, we show that the causal development of the region inside the  $S^{d-2}$ , which we denote  $\mathcal{D}$ , can be mapped to a space  $\mathcal{H} \equiv R \times H^{d-1}$ . Further we show that the vacuum correlators in  $\mathcal{D}$  are conformally mapped to thermal correlators in the space  $\mathcal{H}$ . Hence we are able to show that the density matrix on  $\mathcal{D}$  is given as a Gibbs state of a local operator which is just constructed by conformally mapping the (curved space) Hamiltonian of the CFT in  $\mathcal{H}$  back to  $\mathcal{D}$ . With this construction, the entanglement entropy across the sphere becomes the thermal entropy in  $\mathcal{H}$ . In section 2.3, we demonstrate that the same results apply when we begin with the CFT in a cylindrical background geometry, *i.e.*,  $R \times S^{d-1}$ .

### 2.1 Entanglement entropy in flat space

Consider an arbitrary quantum field theory in  $d$ -dimensional Minkowski space  $R^{1,d-1}$ . As shown in figure 1, we introduce a smooth entangling surface  $\Sigma$  which divides the  $t = 0$  time slice into two parts, a region  $V$  and its complement  $\bar{V}$ . Upon integrating out the degrees of freedom in  $\bar{V}$ , we are left with the reduced density matrix  $\rho$  describing



**Figure 1:** The division of the  $t = 0$  time slice into two regions  $V$  and  $\bar{V}$ .

the remaining degrees of freedom in the region  $V$ . The entanglement entropy across  $\Sigma$  is then just the von Neumann entropy of  $\rho$ , *i.e.*,

$$S_{\Sigma} = -\text{tr}(\rho \log \rho). \quad (2.1)$$

However, we note that the definition and interpretation of these objects in the continuum QFT requires a regularization, as will become apparent below.

Since the reduced density matrix is both hermitian and positive semidefinite, it can be expressed as

$$\rho = e^{-H} \quad (2.2)$$

for some hermitian operator  $H$ . In the literature on axiomatic quantum field theory,  $H$  is known as the modular Hamiltonian [23].<sup>3</sup> We emphasize that generically  $H$  is not a local operator. That is, it can not be represented as some local expression constructed with the fields on  $V$ . However, the modular Hamiltonian still plays an important role since the unitary operator  $U(s) = \rho^{is} = e^{-iHs}$  generates a symmetry of the system. One easily sees that

$$\text{tr}(\rho U(s) \mathcal{O} U(-s)) = \text{tr}(\rho \mathcal{O}), \quad (2.3)$$

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<sup>3</sup>The same operator, referred to as the ‘entanglement Hamiltonian’, has recently also appeared in studies of the ‘entanglement spectrum’ of topological phases of matter [24].

for any operator  $\mathcal{O}$  localized inside  $V$ . In fact, because of causality, this symmetry group transforms the algebra of operators inside the causal development<sup>4</sup> of  $V$  into itself. In the algebraic approach to QFT, this one-parameter group of transformations  $U(s)$  is called the modular group [23]. Further if these transformations are extended to complex parameters, one finds that correlators obey the KMS (Kubo-Martin-Schwinger) periodicity relation in imaginary time. Defining  $\mathcal{O}(s) = U(s)\mathcal{O}U(-s)$ , this relation is easily established with

$$\text{tr}(\rho \mathcal{O}_1(i)\mathcal{O}_2) = \text{tr}(\rho U(i)\mathcal{O}_1 U(-i)\mathcal{O}_2) = \text{tr}(\rho \mathcal{O}_2\mathcal{O}_1). \quad (2.4)$$

The last equality follows using  $U(i) = \rho^{-1}$ ,  $U(-i) = \rho$  and the cyclicity of the trace. Hence formally we can say the state  $\rho$  is thermal with respect to the time evolution dictated by the internal symmetry  $U(s)$ , with an inverse temperature  $\beta = 1$ . However, we emphasize that these are formal expressions. As noted above, generically  $H$  is not local and  $U(s)$  does not generate a local (geometric) flow on  $\mathcal{D}$ . For example, if we begin with a local operator defined at a point,  $\mathcal{O} = \phi(x)$ , then generally the operator  $\mathcal{O}(s)$  will no longer have this simple form of being defined at a point.

However, there are special cases where the modular flow and the modular Hamiltonian are in fact local. One well-known example is given by Rindler space  $\mathcal{R}$ , *i.e.*, the wedge of Minkowski space corresponding to the causal development of the half space  $X^1 > 0$ . In this case for *any* QFT, the modular Hamiltonian is just the boost generator in the  $X^1$  direction. This result is commonly known as the Bisognano-Wichmann theorem [25]. In this case then, the modular transformations act geometrically in Rindler space. They map the algebra of operators  $\mathcal{A}(B)$  localized in a region  $B \subseteq \mathcal{R}$  to the algebra of operators  $\mathcal{A}(B_s)$  in the region  $B_s$ ,

$$U(s)\mathcal{A}(B)U(-s) = \mathcal{A}(B_s), \quad (2.5)$$

where  $B_s$  is the mapping of  $B$  by the boost transformation. More explicitly, the modular flow is given by

$$X^\pm(s) = X^\pm e^{\pm 2\pi s}, \quad (2.6)$$

where  $X^\pm \equiv X^1 \pm X^0$  are the null coordinates with  $0 \leq X^\pm < +\infty$  in  $\mathcal{R}$ . Of course, the modular flow leaves all other coordinates invariant, *i.e.*,  $X^i(s) = X^i$  for  $i = 2, \dots, d-1$ .

Interpreted in the sense of Unruh [26], the state in  $\mathcal{R}$  is thermal with respect to the notion of time translations along the boost orbits. If we choose conventional Rindler coordinates,

$$X^\pm(s) = z e^{\pm \tau/R}, \quad (2.7)$$

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<sup>4</sup>The causal development of  $V$ , which we denote  $\mathcal{D}$ , is the set of all points  $p$  for which all causal curves through  $p$  necessarily intersect  $V$ .

the Minkowski space metric becomes

$$ds^2 = dX^+ dX^- + \sum_{i=2}^{d-1} (dX^i)^2 = -\frac{z^2}{R^2} d\tau^2 + dz^2 + \sum_{i=2}^{d-1} (dX^i)^2 . \quad (2.8)$$

We introduced an arbitrary scale  $R$  above in eq. (2.7) to ensure that  $\tau$  has the standard dimensions of length. With this choice, the Rindler state is thermal with respect to  $H_\tau$ , the Hamiltonian generating  $\tau$  translations, with a temperature  $T = 1/(2\pi R)$ . Hence the density matrix can be simply written as the thermal density matrix

$$\rho_{\mathcal{R}} = \frac{e^{-\beta H_\tau}}{Z} \quad \text{where } Z = \text{tr} (e^{-\beta H_\tau}) . \quad (2.9)$$

With this notation, the modular flow (2.6) on  $\mathcal{R}$  simply corresponds to the time translation

$$\tau \rightarrow \tau + 2\pi R s \quad (2.10)$$

and the modular Hamiltonian  $H_{\mathcal{R}}$  is simply related to  $H_\tau$  with  $H_{\mathcal{R}} = 2\pi R H_\tau + \log Z$ .

In the following, we are particularly interested in the entanglement entropy for the case where the entangling surface  $\Sigma$  is a  $(d-2)$ -dimensional sphere of radius  $R$ . Hence the region  $V$  becomes the ball bounded by this  $S^{d-2}$ . Let us define the null coordinates  $x^\pm \equiv r \pm t$  with  $t = x^0$  and the radial coordinate  $r = \sqrt{(x^1)^2 + \dots + (x^{d-1})^2}$ . Then the causal development  $\mathcal{D}$  of the ball is the spacetime region defined by  $\{x^+ \leq R\} \cap \{x^- \leq R\}$  — implicitly, we are assuming  $x^+ + x^- = 2r \geq 0$  in this definition. Further we wish to consider the special case where the entanglement entropy is calculated in this geometry for a conformal field theory. In this case, the modular Hamiltonian on  $\mathcal{D}$  will in fact be a local operator.

This last fact can be derived making use of the previous result for the Rindler wedge  $\mathcal{R}$ . To begin, we observe that there is a special conformal transformation (and translation) which maps the Rindler wedge to the causal development  $\mathcal{D}$  of the ball (*e.g.*, see [23, 27])

$$x^\mu = \frac{X^\mu - (X \cdot X)C^\mu}{1 - 2(X \cdot C) + (X \cdot X)(C \cdot C)} + 2R^2 C^\mu , \quad (2.11)$$

with  $C^\mu = (0, 1/(2R), 0, \dots, 0)$ . It is straightforward to show that  $X^\pm \geq 0$  covers  $x^\pm \leq R$ . Explicitly, eq. (2.11) yields  $ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = \Omega^2 \eta_{\mu\nu} dx^\mu dx^\nu$  where the conformal pre-factor can be written as

$$\begin{aligned} \Omega &= 1 - 2(X \cdot C) + (X \cdot X)(C \cdot C) . \\ &= (1 + 2(x \cdot C) + (x \cdot x)(C \cdot C))^{-1} . \end{aligned} \quad (2.12)$$



Further eq. (2.11) maps the flow (2.6) to the following geometric flow in  $\mathcal{D}$ ,

$$x^\pm(s) = R \frac{(R + x^\pm) - e^{\mp 2\pi s}(R - x^\pm)}{(R + x^\pm) + e^{\mp 2\pi s}(R - x^\pm)}. \quad (2.13)$$

Now it is not difficult to show that the induced flow (2.13) gives the modular flow of the CFT on  $\mathcal{D}$ . Recall that there is a unitary operator, which we denote  $U_0$ , in the CFT which implements the conformal transformation associated eq. (2.11). For example, the primary operators of the CFT transform locally as (considering spinless operators for simplicity)

$$\phi(x) = \Omega(X)^\Delta U_0 \phi(X) U_0^{-1} \quad (2.14)$$

where  $\Delta$  is the scaling dimension of the field. Since this mapping is a conformal symmetry of Minkowski space, it leaves the vacuum state invariant in a conformal theory, *i.e.*,  $U_0|0\rangle = |0\rangle$ . At a more practical level, vacuum correlators on  $\mathcal{R}$  are mapped to vacuum correlators on  $\mathcal{D}$

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \Omega(X_1)^{\Delta_1} \cdots \Omega(X_n)^{\Delta_n} \langle \phi_1(X_1) \cdots \phi_n(X_n) \rangle. \quad (2.15)$$

Now we may apply  $U_0$  to construct the quantum operator generating the modular flow on  $\mathcal{D}$  with

$$U_{\mathcal{D}}(s) = U_0 U_{\mathcal{R}}(s) U_0^{-1}. \quad (2.16)$$

One can show that  $U_{\mathcal{D}}(s)$  generates the ‘modular transformations’ for the sphere  $\Sigma$  [27]. To confirm that eq. (2.16) yields the modular operator, we must verify that two conditions are satisfied [23]: *i*) the transformation must be a symmetry of the correlators, as in eq. (2.3), and *ii*) the correlators must obey the KMS periodicity, as in eq. (2.4). The first condition eq. (2.3) is evident, since (2.16) is just a particular conformal symmetry of the theory which keeps the sphere invariant.

Turning to the second condition, we note that the correlator on the right hand side of (2.15) satisfies the KMS condition with respect to the flow (2.6). Further combining eqs. (2.6) and (2.12), it is also clear that each of the pre-factors involving  $\Omega$  is also periodic in  $s$  with imaginary period  $i$ . Hence, the correlator on the left hand side also satisfies the KMS condition but with respect to the induced flow (2.13). Hence given that conditions *(i)* and *(ii)* are both satisfied, we conclude that (2.16) are indeed the modular transformations and the geometric flow (2.13) is the modular flow on  $\mathcal{D}$ .

Acting on a (spinless) primary operator in the CFT, one finds

$$U_{\mathcal{D}}(s) \phi(x[s_0]) U_{\mathcal{D}}(-s) = \Omega(x[s_0])^\Delta \Omega(x[s_0 + s])^{-\Delta} \phi(x[s_0 + s]), \quad (2.17)$$

using eq. (2.14). Here, our notation  $x[s_0]$  indicates that the position of the operator flows according to eq. (2.13). Note then that  $U_{\mathcal{D}}(s)$  acts both to translate the operator

along the geometric flow (2.13) and multiplies it with a particular (c-number) pre-factor. The key point, however, is that  $U_{\mathcal{D}}(s)$  remains a local transformation, taking an operator defined at a point to that same operator defined at a new point.

In fact, we can produce an explicit expression for the modular Hamiltonian at this point. The construction is simplest if we focus on  $V$ , the ball of radius  $R$ , in the time slice  $x^0 = 0$ . Since the modular Hamiltonian is the generator for the transformation (2.17), it will produce an infinitesimal shift  $\delta s$  on the surface  $x^0 = 0$ :

$$\delta x^0 = 2\pi \frac{R^2 - r^2}{2R} \delta s \quad \text{and} \quad \delta r = 0. \quad (2.18)$$

At the same time, an infinitesimal shift  $\delta s$  produces a pre-factor in eq. (2.17) which takes the form

$$\Omega(x[s_0])^\Delta \Omega(x[s_0 + \delta s])^{-\Delta} \Big|_{x^0=0} \simeq 1 - \Delta \left. \frac{\partial_s \Omega}{\Omega} \right|_{x^0=0} \delta s. \quad (2.19)$$

However, using eqs. (2.12) and (2.18), it is straightforward to show that  $\partial_s \Omega = 0$  when evaluated on the  $x^0 = 0$  surface and so the pre-factor is simply 1 at this order. Hence the modular Hamiltonian simply induces the infinitesimal flow (2.18) away from  $x^0 = 0$  and we can identify the corresponding operator in the CFT as

$$H_{\mathcal{D}} = 2\pi \int_V d^{d-1}x \frac{(R^2 - r^2)}{2R} T^{00}(x) + c', \quad (2.20)$$

where  $T^{\mu\nu}$  is the conformal *traceless* stress tensor and  $c'$  is some constant added to ensure that the corresponding density matrix is normalized with unit trace. Hence this expression explicitly shows the modular Hamiltonian as a local operator, for a CFT in the causal development  $\mathcal{D}$ .

One point that we feel is worth emphasizing is that the standard CFT correlators in  $\mathcal{D}$  transform ‘covariantly’ under  $U_{\mathcal{D}}(s)$  — that is, they are *not* invariant, as one might naïvely surmise from eq. (2.3). This is simply the observation that if we begin with  $\mathcal{O} = \phi_1(x_1[s_0]) \cdots \phi_n(x_n[s_0])$  in eq. (2.3), then eq. (2.17) dictates that generally  $\mathcal{O}(s) \neq \phi_1(x_1[s_0 + s]) \cdots \phi_n(x_n[s_0 + s])$ . Instead, the correlators of (spinless) primary operators transform as

$$\begin{aligned} & \Omega(x_1[s_0])^{-\Delta_1} \cdots \Omega(x_n[s_0])^{-\Delta_n} \langle \phi_1(x_1[s_0]) \cdots \phi_n(x_n[s_0]) \rangle \\ &= \Omega(x_1[s_0 + s])^{-\Delta_1} \cdots \Omega(x_n[s_0 + s])^{-\Delta_n} \langle \phi_1(x_1[s_0 + s]) \cdots \phi_n(x_n[s_0 + s]) \rangle. \end{aligned} \quad (2.21)$$

Clearly, the idea of ‘transplanting’ the modular flow with a conformal transformation can be used to obtain the modular Hamiltonian (and the density matrix) for a CFT

in other conformally connected geometries. Note that the state in the transformed space has to be chosen as the conformally transformed state, which is the one that makes (2.21) work. In the present case of a mapping between the Rindler wedge  $\mathcal{R}$  and the causal development  $\mathcal{D}$ , the transformation is a conformal symmetry of Minkowski space and the transformed state is again the Minkowski vacuum. Of course, for operators inside  $\mathcal{D}$ , the vacuum coincides with the density matrix  $\rho$ , meaning that they both give the same expectation values  $\langle 0|\mathcal{O}|0\rangle = \text{tr}(\rho\mathcal{O})$ .

## 2.2 Thermal behaviour in $R \times H^{d-1}$

Next we would like to extend this approach of transplanting modular flows to relate the density matrix of a CFT on  $\mathcal{D}$  to that on a new geometry  $R \times H^{d-1}$ , which we will denote as  $\mathcal{H}$  in the following. In particular, we will show that beginning with the Minkowski vacuum for an arbitrary CFT, the density matrix on  $\mathcal{D}$  becomes a thermal density matrix on  $\mathcal{H}$ . In fact, our result is a generalization of previous observations made in ref. [16]. There the generation of a thermal state by conformal mappings was observed for free conformal field theories in  $d = 4$ .

One indication that the claim above holds comes from first returning to the Rindler wedge (2.8). Here, we can write the metric as

$$ds^2 = \Omega^2 \left( -d\tau^2 + \frac{R^2}{z^2} \left[ dz^2 + \sum_{i=2}^{d-1} (dX^i)^2 \right] \right), \quad (2.22)$$

where  $\Omega = z/R$ . Hence with a conformal transformation which eliminates the pre-factor  $\Omega^2$ , the Rindler metric is mapped precisely to the metric on  $R \times H^{d-1}$ . Note that the scale  $R$  now sets the curvature scale of the hyperbolic plane  $H^{d-1}$ . As in the previous section, there is a unitary operator which maps the CFT from  $\mathcal{R}$  to  $\mathcal{H}$ , which we will denote  $U_1$ . Further we may again apply  $U_1$  to construct the operator generating the modular flow on  $\mathcal{H}$  with

$$U_{\mathcal{H}}(s) = U_1 U_{\mathcal{R}}(s) U_1^{-1}. \quad (2.23)$$

In this particular case, the conformal mapping is time (*i.e.*,  $\tau$ ) independent and so  $U_1$  acts ‘trivially’ on the modular Hamiltonian. That is,  $H_{\mathcal{H}} = 2\pi R H_{\tau} + \log Z$  where  $H_{\tau}$  is now the generator of  $\tau$ -translations in  $\mathcal{H}$ . Therefore the new density matrix  $\rho_{\mathcal{H}}$  inherits the same thermal character as in eq. (2.9). Hence combining this result with those in the previous section, we can map the reduced density matrix on  $\mathcal{D}$  to a thermal density matrix on  $\mathcal{R}$  and then to a thermal density matrix on  $\mathcal{H}$ . However, let us step back and establish the relationship between  $\rho_{\mathcal{D}}$  and  $\rho_{\mathcal{H}}$  directly.

First, we present the conformal transformation which maps  $\mathcal{D}$  to  $\mathcal{H}$ . We start with the flat space metric in polar coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2, \quad (2.24)$$

where  $d\Omega_{d-2}^2$  is the line element on a round  $S^{d-2}$  with unit curvature. Our entangling surface  $\Sigma$  is again the sphere  $r = R$  on the surface  $t = 0$ . Now we make the coordinate transformation

$$\begin{aligned} t &= R \frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}, \\ r &= R \frac{\sinh u}{\cosh u + \cosh(\tau/R)}. \end{aligned} \quad (2.25)$$

One can readily verify the above metric (2.24) becomes

$$\begin{aligned} ds^2 &= \Omega^2 \left[ -d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2) \right] \\ \text{where } \Omega &= (\cosh u + \cosh(\tau/R))^{-1}. \end{aligned} \quad (2.26)$$

After the conformal transformation which eliminates the pre-factor  $\Omega^2$ , we again recognize the resulting line element as the metric on  $R \times H^{d-1}$ . The curvature on the latter hyperbolic space is

$$R_{ij}{}^{k\ell} = -\frac{1}{R^2} (\delta_i^k \delta_j^\ell - \delta_i^\ell \delta_j^k). \quad (2.27)$$

We note that fixing the curvature scale to match  $R$ , the radius of the sphere  $\Sigma$ , is an arbitrary but convenient choice. Finally we observe that

$$\begin{aligned} \tau \rightarrow \pm\infty : \quad & (t, r) \rightarrow (\pm R, 0) \\ u \rightarrow \infty : \quad & (t, r) \rightarrow (0, R) \end{aligned}$$

Hence the new coordinates cover precisely  $\mathcal{D}$ , the causal development of the region inside of  $\Sigma$ . Hence we have our conformal mapping from  $\mathcal{D}$  to  $\mathcal{H}$ . We are again considering an arbitrary CFT and so there is a unitary transformation  $U_2$  implementing the conformal transformation (2.25) on the Hilbert space of the CFT.<sup>5</sup>

Now, we wish to relate the two reduced density matrices:  $\rho_D$  describing the vacuum state on  $\mathcal{D}$  and  $\rho_{\mathcal{H}}$  for the corresponding state on  $\mathcal{H}$ . In particular, we wish to establish that the latter corresponds to a thermal density matrix with temperature  $T = 1/(2\pi R)$ . For the latter to hold, we must verify two conditions: First, the modular flow in  $\mathcal{H}$  must correspond to ordinary time translations and these translations must be a symmetry of

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<sup>5</sup>Of course, this operator is related to those appearing in the previous discussion by  $U_2 = U_1 U_0^{-1}$ .

the correlators. Second, the correlators on  $\mathcal{H}$  must be periodic for an imaginary shift of  $\tau$  by  $2\pi R$ .

As a first step, we use eq. (2.25) to write

$$x^\pm = R \frac{1 - e^{-v^\pm}}{1 + e^{-v^\pm}} \quad (2.28)$$

where we have defined  $v^\pm \equiv u \pm (\tau/R)$  — recall from above that  $x^\pm = r \pm t$ . From these expressions, it is straightforward to show that the modular flow (2.13) on  $\mathcal{D}$  corresponds to a time translation

$$\tau \rightarrow \tau + 2\pi R s \quad (2.29)$$

in  $\mathcal{H}$  — just as in eq. (2.10). Hence as desired, the modular Hamiltonian induces a flow along time translations. However, for a thermal state, this flow must correspond to a symmetry of the correlators, *i.e.*, it is not enough that the correlators transform covariantly under the modular flow. The primary fields transform by the analog of eq. (2.14) replacing  $U_0$  by  $U_2$  and hence we can write the relation for the correlation functions under the conformal transformation as

$$\langle \phi'_1(x'_1) \cdots \phi'_n(x'_n) \rangle_{\mathcal{H}} = \Omega(x'_1)^{\Delta_1} \cdots \Omega(x'_n)^{\Delta_n} \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle_{\mathcal{D}}. \quad (2.30)$$

where  $\Omega(x')$  is the conformal pre-factor in eq. (2.26)). Now the factors  $\Omega$  in (2.30) are not invariant under shifts of  $\tau$ . However, we also noted above in eq. (2.21) that the modular flow on  $\mathcal{D}$  transforms the correlators on the right-hand side non-trivially. Having the connection (2.29) between translations in  $\tau$  and  $s$ , a simple calculation shows the  $\tau$  dependence of the pre-factors precisely cancels that of the correlators on  $\mathcal{D}$ . Hence the conformally mapped correlators on  $\mathcal{H}$  are in fact invariant under shifts in  $\tau$ . Hence the modular flow in  $\mathcal{H}$  is just the ordinary time translations (2.29). Examining eq. (2.30) further, we find the second requirement, that the correlators on  $\mathcal{H}$  must be periodic for an imaginary shift of  $\tau$  by  $2\pi R$  also follows. First, given eq. (2.26), it is clear that each of conformal pre-factors satisfies this condition. Next the correlators on  $\mathcal{D}$  satisfy the KMS condition (2.4) and from eq. (2.29), it follows that the imaginary shift of  $s$  by  $i$  corresponds to the desired shift of  $\tau$ .

Hence the two requirements above are both satisfied and so we have established the desired result: The conformal mapping (2.26) takes the CFT in the Minkowski vacuum on  $\mathcal{D}$  to a thermal state on  $\mathcal{H}$  where the latter is thermal with respect to the standard Hamiltonian  $H_\tau$  with a physical temperature  $T = 1/(2\pi R)$ . That is, we have  $\rho_{\mathcal{H}} = e^{-\beta H_\tau}/Z$ . Now the unitary operator  $U_2$  will map between the reduced density matrix on  $\mathcal{D}$  and the thermal density matrix on  $\mathcal{H}$ . More explicitly, we may write the density matrix on  $\mathcal{D}$  as

$$\rho_{\mathcal{D}} \equiv e^{-H_{\mathcal{D}}} = \frac{1}{Z} U_2^{-1} e^{-\beta H_\tau} U_2. \quad (2.31)$$

Since the von Newman entropy is invariant under unitary transformations, the entanglement entropy across the sphere  $\Sigma$  in flat space is then equal to the corresponding thermal entropy of the Gibbs state in  $\mathcal{H}$ .

However, some care must be exercised for the equality of the entropies to hold since both of these quantities are divergent. In particular, in order to maintain the equality, we must also use the conformal transformation to map between the cut-off procedures in two spaces. In the case of entanglement entropy (2.1), there is a UV divergence at the boundary  $\Sigma$  and so we need to introduce a short distance cut-off scale  $\delta$ . That is, we only take contributions down to  $r = R - \delta$  where  $\delta/R \ll 1$ .<sup>6</sup> In the case of the thermal entropy, there is an IR divergence because we have a uniform entropy density but the volume of the spatial slices, *i.e.*,  $H^{d-1}$ , is infinite. Hence we regulate the entropy by integrating out to some maximum radius,  $u = u_{max}$  where  $u_{max} \gg 1$ . Now, consistency demands that the cut-off's should be related by the conformal mapping between the two spaces. If we focus on the  $t = 0$  slice (or equivalently the  $\tau = 0$  slice), then eq. (2.25) yields

$$R - \delta = R \frac{\sinh u_{max}}{\cosh u_{max} + 1}. \quad (2.32)$$

After a bit of algebra, we find

$$\exp(-u_{max}) = \frac{\delta/R}{2 - \delta/R} \simeq \frac{\delta}{2R}. \quad (2.33)$$

It is interesting that here the conformal mapping has introduced a UV/IR relation between the relevant cut-offs in the two CFT states. Of course, similar relations commonly occur in the AdS/CFT correspondence but here we have a purely CFT calculation.

### 2.3 Entanglement entropy in a cylindrical background

In this section, we develop the analogous account of the ‘thermalization by conformal mapping’ found above for the case where the original background geometry is  $R \times S^{d-1}$ . That is, we wish to calculate the entanglement entropy across a sphere of a fixed angular size embedded in the  $t = 0$  slice of the static Einstein universe. As before, we map the causal development of the interior of this sphere to  $R \times H^{d-1}$  with a conformal transformation and show that the resulting density matrix describes a thermal state. Hence the entanglement entropy becomes the thermal entropy of the Gibbs state in  $\mathcal{H}$ . Below we only present the salient features of the proof as conceptually it is completely analogous to the discussion above.

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<sup>6</sup>A regularization of the entanglement entropy implementing a distance cutoff to the entangling sphere can be made precise using the mutual information [28].

We start with the metric in  $d$ -dimensional cylindrical spacetime with topology  $R \times S^{d-1}$ :

$$ds^2 = -dt^2 + R^2 (d\theta^2 + \sin^2\theta d\Omega_{d-2}^2) . \quad (2.34)$$

The entangling surface  $\Sigma$  will now be the sphere  $\theta = \theta_0$  on the surface  $t = 0$ . Now make the coordinate transformation

$$\begin{aligned} \tan(t/R) &= \frac{\sin\theta_0 \sinh(\tau/R)}{\cosh u + \cos\theta_0 \cosh(\tau/R)} , \\ \tan\theta &= \frac{\sin\theta_0 \sinh u}{\cos\theta_0 \cosh u + \cosh(\tau/R)} . \end{aligned} \quad (2.35)$$

One can readily verify the above metric (2.34) becomes

$$\begin{aligned} ds^2 &= \Omega^2 [-d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2)] \\ \text{where } \Omega^2 &= \frac{\sin^2\theta_0}{(\cosh u + \cos\theta_0 \cosh(\tau/R))^2 + \sin^2\theta_0 \sinh^2 u} . \end{aligned} \quad (2.36)$$

Hence, after eliminating the conformal factor  $\Omega^2$  in the first line, we again recognize the final line element as the metric on  $R \times H^{d-1}$ ,

$$ds'^2 = -d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2) , \quad (2.37)$$

precisely as in eq. (2.26). The curvature scale of the hyperbolic space is  $R$ , as in eq. (2.27). As before, we note that this is an arbitrary but convenient choice for the curvature. In this case,  $R$  also corresponds to the radius of curvature of the  $S^{d-1}$  in the Einstein universe, rather than the size of the entangling sphere  $\Sigma$ , as we will see below.

Examining what portion of the original Einstein universe (2.34) is covered by the new coordinates, we find

$$\begin{aligned} \tau \rightarrow \pm\infty : & \quad (t, \theta) \rightarrow (\pm R\theta_0, 0) \\ u \rightarrow \infty : & \quad (t, \theta) \rightarrow (0, \theta_0) \end{aligned}$$

Hence, the new coordinates cover precisely the causal development  $\mathcal{D}$  of the ball enclosed by the entangling sphere  $\Sigma$ . As in section 2.3, examining the effect of the conformal mapping on CFT correlators, we find that the vacuum correlators in  $\mathcal{D}$  become thermal correlators on  $\mathcal{H}$  with temperature  $T = 1/(2\pi R)$ .

As before, the entanglement entropy across the sphere  $\Sigma$  in the Einstein space matches the thermal entropy of the Gibbs state in  $\mathcal{H}$  but we must be careful in regulating the two expressions. In the case of entanglement entropy, there is a UV divergence

at the surface  $\Sigma$  and we need to introduce a short distance cut-off. The natural cut-off is a minimal angular size  $\delta\theta$  ( $\ll 1$ ) and only taking contributions out to  $\theta = \theta_0 - \delta\theta$ . As before, the thermal entropy has an IR divergence because we have a uniform entropy density in the infinite volume of the spatial  $H^{d-1}$  slices. We again regulate this entropy by only integrating out to  $u = u_{max}$  where  $u_{max} \gg 1$ . Now consistency demands that the cut-off's should be related by the conformal mapping between  $\mathcal{D}$  and  $\mathcal{H}$ . If we focus on the  $t = 0$  slice (or equivalently the  $\tau = 0$  slice), then eq. (2.35) yields

$$\tan(\theta_0 - \delta\theta) = \frac{\sin \theta_0 \sinh u_{max}}{\cos \theta_0 \cosh u_{max} + 1}. \quad (2.38)$$

With a bit of algebra, we find

$$\exp(-u_{max}) = \frac{1}{2 \sin \theta_0} \frac{\tan \delta\theta + \tan \theta_0 (\sec \delta\theta - 1)}{1 - \cot 2\theta_0 \tan \delta\theta} \simeq \frac{\delta\theta}{2 \sin \theta_0}. \quad (2.39)$$

Hence as in the previous section, we have an interesting UV/IR relation between the cut-offs in the two CFT calculations.

### 3. The AdS story

In the previous section, we related the problem of calculating entanglement entropy for a spherical entangling surface to calculating the thermal entropy of a Gibbs state in  $R \times H^{d-1}$ , where the temperature is of the same order as the curvature scale of the hyperbolic geometry. While this is quite general result that applies for any arbitrary CFT, it seems that we have only replaced one difficult problem with another equally difficult problem. However, we are particularly interested in applying this result to the AdS/CFT correspondence. In this framework, the standard holographic dictionary suggests that the thermal state in the boundary CFT is dual to a black hole in the bulk gravity theory. Hence if we are able to identify the corresponding black hole, the thermal entropy of the CFT can be calculated as the horizon entropy of the black hole.

Precisely this problem was encountered in [13, 14] in a related calculation of the entanglement entropy for a specific geometry. In this case, the boundary CFT was placed in a cylindrical space,  $R \times S^{d-1}$  — that is, the (complete) boundary of  $\text{AdS}_{d+1}$ . The problem was then to determine the entanglement entropy when the  $(d-1)$ -dimensional sphere was divided in half along the equator, *i.e.*, the entangling surface was chosen as a maximal  $S^{d-2}$  in a constant time slice. In [13, 14], it was argued that the entanglement entropy could be identified with the horizon entropy of a topological AdS black hole, for which the large radius limit (at fixed time) covered precisely one half of the boundary  $S^{d-1}$ . In fact, the topological black hole was simply an  $R \times H^{d-1}$  foliation of the  $\text{AdS}_{d+1}$  geometry.



Given the CFT discussion of section 2, we now have a clearer understanding of the calculations in [13, 14] and in turn, the calculations there suggest the necessary approach to implement our results from the previous section in a holographic setting. In particular, we begin with the boundary CFT in its vacuum either in Minkowski space  $R^{1,d-1}$  or the cylindrical space  $R \times S^{d-1}$ . The dual gravity description is simply the pure  $\text{AdS}_{d+1}$  geometry in a coordinate system which foliates the spacetime with  $R^{1,d-1}$  or  $R \times S^{d-1}$  surfaces. To calculate the entanglement entropy of the CFT across some sphere  $\Sigma$  in the boundary geometry, we must next find a new foliation of the  $\text{AdS}_{d+1}$  in terms of  $R \times H^{d-1}$ . This foliation is chosen to cover the ball enclosed by  $\Sigma$  in the asymptotic boundary geometry. In fact, the causal development  $\mathcal{D}$  of this ball will be covered in the asymptotic limit of the  $R \times H^{d-1}$  foliation. Implicitly, we will have implemented the conformal mapping of the causal development  $\mathcal{D}$  of the ball to  $R \times H^{d-1}$  in the boundary CFT with the bulk coordinate transformation between the two foliations of the AdS space. As noted in [13, 14], the new hyperbolic foliation can be interpreted in terms of a topological black hole [29, 30] and so on the  $R \times H^{d-1}$  background, the boundary CFT is naturally seen to be in a thermal state. The temperature of the latter thermal state is given by the Hawking temperature of the black hole horizon and we will find that we precisely reproduce the result of section 2:  $T = 1/(2\pi R)$ . Further, since as discussed in section 2, the entanglement entropy across the sphere  $\Sigma$  can be calculated as the thermal entropy of the Gibbs state in the  $R \times H^{d-1}$  geometry, the same entropy is given by the horizon entropy of the bulk black hole in the holographic setting.

Let us emphasize that this approach gives us a derivation of the entanglement entropy of the holographic CFT, in the case of a spherical entangling surface. Given the insights of the previous section, we are simply applying the standard AdS/CFT dictionary to implement the results there in a holographic framework. Further let us note that if the bulk theory was Einstein gravity, then our results of the entanglement entropy for this particular set of geometries would precisely match the holographic entanglement entropy calculated with the extremal area prescription (1.1) conjectured by Ryu and Takayanagi [4, 5]. In this regard, our results provide a nontrivial confirmation of their proposal.

In fact, since we have a derivation of the entanglement entropy, we do not need to limit our discussion to Einstein gravity. Hence in the following discussion, we allow the bulk gravity to be described by any arbitrary covariant action of the form

$$I = \int d^{d+1}x \sqrt{-g} \mathcal{L}(g^{ab}, R^{ab}_{cd}, \nabla_e R^{ab}_{cd}, \dots, \text{matter}). \quad (3.1)$$

Hence one might apply our analysis in the context of the low energy effective action

in string theory, where the contributions of higher curvature terms are controlled to make small corrections to the results of the Einstein theory [31]. Alternatively, our results would also be applicable to a situation where higher curvature contributions are finite, as in the recent studies of the AdS/CFT correspondence [32] with Lovelock [33] or quasi-topological [34] gravity. In the following, we will assume that the couplings of the above theory are chosen so that the theory has a vacuum solution which is  $\text{AdS}_{d+1}$  spacetime with a curvature scale  $L$ . Implicitly, we will also assume that the bulk couplings are further constrained so that the boundary CFT has physically reasonable properties (*e.g.*, it should be causal and unitary) — for example, see [14, 34, 32].

An essential step in the following will be calculating the horizon entropy of the bulk black hole. In general, the horizon entropy can be calculated using Wald’s entropy formula [35]

$$S = -2\pi \int_{\text{horizon}} d^{d-1}x \sqrt{h} \frac{\partial \mathcal{L}}{\partial R^{ab}_{cd}} \hat{\varepsilon}^{ab} \hat{\varepsilon}_{cd}, \quad (3.2)$$

which can be applied for any (covariant) action, as assumed above in eq. (3.1). Note that  $\hat{\varepsilon}_{ab}$  denotes the binormal to the horizon. Of course, as described above, the case of interest is a topological black hole which in fact corresponds to a  $R \times H^{d-1}$  foliation of  $\text{AdS}_{d+1}$ . In this case, the integrand in eq. (3.2) is constant across the horizon and so the total entropy diverges. Of course, this divergence is expected given the discussion towards the end of section 2.2 and we will return to this point in the following. However, at this point, we use some results from [14] to re-express the integrand as

$$\left. \frac{\delta \mathcal{L}}{\delta R^{ab}_{cd}} \hat{\varepsilon}^{ab} \hat{\varepsilon}_{cd} \right|_{\text{AdS}} = -\frac{\Gamma(d/2)}{\pi^{d/2}} \frac{a_d^*}{L^{d-1}} \quad (3.3)$$

where  $L$  is the AdS curvature scale. The (dimensionless) constant  $a_d^*$  is a central charge that characterizes the number of degrees of freedom in the boundary CFT [14]. In the case where  $d$ , the dimension of boundary theory, is even,  $a_d^*$  is precisely equal to  $A$ , the coefficient of the  $A$ -type trace anomaly (1.2) in the CFT [14]. We stress that eq. (3.3) relies on the fact that the background geometry in which we are evaluating this expression is simply the  $\text{AdS}_{d+1}$  spacetime. Substituting this result into eq. (3.2) leaves us with

$$S = \frac{2\Gamma(d/2)}{\pi^{d/2-1}} \frac{a_d^*}{L^{d-1}} \int_{\text{horizon}} d^{d-1}x \sqrt{h}. \quad (3.4)$$

We now turn to the detailed determination of the hyperbolic foliations discussed above. We begin by examining the entangling surface being a sphere in flat space in the next section and then consider the case with a cylindrical background in section 3.2.

### 3.1 Entanglement entropy in flat space

Here we want to present a holographic calculation that implements the discussion of entanglement entropy of a CFT in flat space given in section 2.1. So let us begin with a standard description of the  $\text{AdS}_{d+1}$  geometry as the following surface

$$-y_{-1}^2 - y_0^2 + y_1^2 + \cdots + y_d^2 = -L^2 \quad (3.5)$$

embedded in  $R^{2,d}$  with

$$ds^2 = -dy_{-1}^2 - dy_0^2 + dy_1^2 + \cdots + dy_d^2. \quad (3.6)$$

Now we can connect this geometry to the standard Poincaré coordinates on  $\text{AdS}_{d+1}$  space with

$$y_{-1} + y_d = \frac{L^2}{z}, \quad y^a = \frac{L}{z} x^a \quad \text{with } a = 0, \dots, d-1. \quad (3.7)$$

Now eq. (3.5) becomes a constraint which yields

$$y_{-1} - y_d = z + \frac{1}{z} \eta_{ab} x^a x^b. \quad (3.8)$$

The induced metric on the hyperboloid (3.5) then becomes the standard  $\text{AdS}_{d+1}$  metric

$$ds^2 = \frac{L^2}{z^2} (dz^2 + \eta_{ab} dx^a dx^b) \quad (3.9)$$

where the bulk spacetime is foliated with slices corresponding to flat  $d$ -dimensional Minkowski space. As usual in the AdS/CFT framework, we take the asymptotic limit  $z \rightarrow 0$  and remove a factor of  $L^2/z^2$  from the boundary metric. This yields  $ds_{\text{CFT}}^2 = \eta_{ab} dx^a dx^b$  as the metric in which the dual CFT lives when making our holographic calculations.

Another useful foliation of  $\text{AdS}_{d+1}$  is given by

$$y_{-1} = \rho \cosh u, \quad y_0 = \tilde{\rho} \sinh(\tilde{\tau}/L), \quad y_d = \tilde{\rho} \cosh(\tilde{\tau}/L), \quad (3.10)$$

$$y_1 = \rho \sinh u \cos \phi_1, \quad y_2 = \rho \sinh u \sin \phi_1 \cos \phi_2, \dots, y_{d-1} = \rho \sinh u \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{d-2}.$$

In this case, the constraint imposed by eq. (3.5) yields  $\tilde{\rho}^2 = \rho^2 - L^2$  and the induced metric on the hyperboloid (3.5) becomes

$$ds^2 = \frac{d\rho^2}{\frac{\rho^2}{L^2} - 1} - \left( \frac{\rho^2}{L^2} - 1 \right) d\tilde{\tau}^2 + \rho^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2) \quad (3.11)$$

where  $d\Omega_{d-2}^2 = d\phi_1^2 + \sin^2 \phi_1 (d\phi_2^2 + \sin^2 \phi_2 (d\phi_3^2 + \cdots))$  is the line element on a unit  $S^{d-2}$ . Of course, the bracketed expression in eq. (3.11), which is multiplied by  $\rho^2$ ,

corresponds to the line element on a  $(d-1)$ -dimensional hyperbolic plane  $H^{d-1}$  with unit curvature. Hence as desired, we are foliating the  $\text{AdS}_{d+1}$  space with surfaces with a  $R \times H^{d-1}$  geometry in the above metric (3.11). Taking the asymptotic limit and removing a factor of  $\rho^2/L^2$ , we find the boundary metric is precisely that given in eq. (2.37) with  $R = L$  and  $\tilde{\tau} = \tau$ .

For our purposes, the essential feature of the second metric (3.11) is that it can be interpreted as a topological black hole [29, 30] where the horizon at  $\rho = L$  has uniform negative curvature. We would like to see what portion of the asymptotic AdS boundary in the Poincaré coordinates is covered by the exterior of this topological black hole. We begin by considering the bifurcation surface (*i.e.*, the intersection of the past and future horizons) in eq. (3.11) which corresponds to  $\rho = L$  and any finite value of  $\tilde{\tau}$ . Combining eqs. (3.7) and (3.10), we find on this surface

$$y_{-1} + y_d = \frac{L^2}{z} = L \cosh u, \quad y_1^2 + \dots + y_{d-1}^2 = \frac{L^2}{z^2} r^2 = L^2 \sinh^2 u. \quad (3.12)$$

where we have introduced a radial coordinate for the boundary coordinates, *i.e.*,  $r^2 = (x^1)^2 + \dots + (x^{d-1})^2$ . Hence we can see that the bifurcation surface intersects the asymptotic boundary in the Poincaré coordinates with

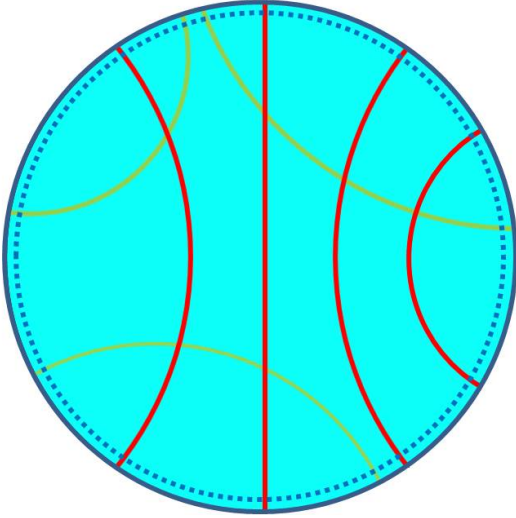
$$r^2 = z^2 \sinh^2 u = L^2 \tanh^2 u \xrightarrow{u \rightarrow \infty} L^2. \quad (3.13)$$

That is, on a sphere of radius  $r = L$  in the Minkowski coordinates of the boundary CFT. This will be the entangling sphere  $\Sigma$  in the calculation of the entanglement entropy. One can work harder to find the precise connection between the two coordinate systems on the boundary of  $\text{AdS}_{d+1}$ . The resulting coordinate transformation is precisely that given in eq. (2.25) with  $R = L$  and  $\tilde{\tau} = \tau$ .<sup>7</sup>

Next we must determine the Hawking temperature of our topological black hole (3.11). A straightforward calculation shows that  $T = 1/(2\pi L)$ . Given that the entangling sphere has a radius  $R = L$ , this result precisely matches the temperature found in section 2, where the discussion applied for any CFT without reference to holography. Hence as discussed in the introductory remarks of this section, changing coordinates in the bulk between the Poincaré and hyperbolic foliations implements the conformal mapping from  $\mathcal{D}$ , the causal development of the sphere at  $r = L$ , to  $R \times H^{d-1}$ . As expected from the general considerations of section 2, this mapping generates a thermal state in the  $R \times H^{d-1}$  geometry. The entanglement entropy across the sphere  $r = L$  in flat space equals the thermal entropy of this Gibbs state and further, in the holographic setting, the latter is given by the horizon entropy of the bulk black hole.

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<sup>7</sup>In fact, examining the asymptotic relation between the Poincaré and hyperbolic foliations is how we originally derived the transformation (2.25).



**Figure 2:** A slice of constant  $t$  through the  $\text{AdS}_{d+1}$  spacetime. The solid (red and green) arcs each represent codimension two surfaces with constant curvature  $-1/L^2$ . These are all connected by isometries of the  $\text{AdS}_{d+1}$  geometry. The red arcs are connected by a boost, as described in the main text. The dashed blue line represents a regulator surface at large radius.

low, we will see that the regulator surface plays an essential role in the calculation of the entanglement entropy. This surface is typically not invariant under the isometries that connect the various candidate bifurcation surfaces and consequently, *e.g.*, the entropy will be different with different choices.

One simple isometry relating different bifurcation surfaces, illustrated in figure 2, is a boost in the embedding space of the  $y_A$  coordinates. In particular, a boost in the  $(y_{-1}, y_d)$ -plane leaves the hyperboloid (3.5) invariant while transforming the coordinates:

$$\begin{aligned} y'_{-1} &= \cosh \beta y_{-1} - \sinh \beta y_d, \\ y'_d &= \cosh \beta y_d - \sinh \beta y_{-1}. \end{aligned} \tag{3.14}$$

Now we make the trivial substitution to replace eq. (3.10) with

$$\begin{aligned} y'_{-1} &= \rho \cosh u, \quad y_0 = \tilde{\rho} \sinh(\tilde{\tau}/L), \quad y'_d = \tilde{\rho} \cosh(\tilde{\tau}/L), \\ y_1 &= \rho \sinh u \cos \phi_1, \quad y_2 = \rho \sinh u \sin \phi_1 \cos \phi_2, \dots y_{d-1} = \rho \sinh u \sin \phi_1 \sin \phi_2 \dots \sin \phi_{d-2}. \end{aligned} \tag{3.15}$$

We leave the calculation of the entropy for a bit later. First we would like to extend the previous holographic calculations to a sphere in Minkowski space with an arbitrary radius  $R$ , as in eq. (2.25). Essentially we want to move the bifurcation surface above to a new position that intersects the boundary in a different way, as illustrated in figure 2. An interesting observation is that in fact all of the surfaces illustrated in this figure are ‘identical’ in that they are connected by an isometry of the  $\text{AdS}_{d+1}$  geometry. Hence one finds that the bulk metric for the hyperbolic foliation in which any of these surfaces appears as the bifurcation surface is identical to that in eq. (3.11). Given these isometries, one might ask how the physics (*e.g.*, the entropy) can change. However, in the AdS/CFT framework, calculations are always made with reference to a regulator surface in the asymptotic region, *e.g.*,  $z = z_{\min}$  in the Poincaré metric (3.9). Be-

As before, the constraint (3.5) yields  $\tilde{\rho}^2 = \rho^2 - L^2$  and the induced metric is identical to that given in eq. (3.11). We leave our choice of Poincaré coordinates unchanged as in eq. (3.7). Hence to relate the two coordinate patches, it is useful to write

$$\begin{aligned} y_{-1} &= \cosh \beta \rho \cosh u + \sinh \beta \sqrt{\rho^2 - L^2} \cosh(\tilde{\tau}/L), \\ y_d &= \cosh \beta \sqrt{\rho^2 - L^2} \cosh(\tilde{\tau}/L) + \sinh \beta \rho \cosh u. \end{aligned} \quad (3.16)$$

Again, this hyperbolic foliation (3.11) lends itself to an interpretation as a topological black hole. However, the horizon has been displaced relative to the Poincaré coordinate patch. To understand the relation between the two coordinate patches, we examine how the bifurcation surface at  $\rho = L$  (and any finite value of  $\tilde{\tau}$ ) approaches the AdS boundary. Combining eqs. (3.7) and (3.16), we find

$$y_{-1} + y_d = \frac{L^2}{z} = e^\beta L \cosh u, \quad y_1^2 + \dots + y_{d-1}^2 = \frac{L^2}{z^2} r^2 = L^2 \sinh^2 u. \quad (3.17)$$

Recall that  $r^2 = (x^1)^2 + \dots + (x^{d-1})^2$ . Hence we can see that the bifurcation surface intersects the asymptotic boundary in the Poincaré coordinates with

$$r^2 = z^2 \sinh^2 u = e^{-2\beta} L^2 \tanh^2 u \xrightarrow{u \rightarrow \infty} e^{-2\beta} L^2. \quad (3.18)$$

That is, on a sphere of radius  $R = e^{-\beta} L$  in the flat boundary metric. With a bit of work, one can determine the precise relation between the two coordinate systems on the boundary of  $\text{AdS}_{d+1}$ . In fact, one finds precisely the coordinate transformation (2.25) with  $R = e^{-\beta} L$  and  $\tilde{\tau} = e^{-\beta} \tau$ .

Interpreting the bulk metric (3.11) as a topological black hole, the boundary CFT on  $R \times H^{d-1}$  is in a thermal state. Since the metric is unchanged, the Hawking temperature of the black hole will be  $\tilde{T} = 1/(2\pi L)$ , precisely as before. We denote this temperature as  $\tilde{T}$  since it is determined for energies conjugate to time translations in  $\tilde{\tau}$ . To compare to the discussion in section 2, we wish to determine the temperature  $T$  conjugate to the time coordinate  $\tau$  in the metric (2.37). As noted above, these two coordinates are related by the scaling  $\tilde{\tau} = e^{-\beta} \tau$ . Hence the temperatures are also related by  $\tilde{T} = e^{-\beta} T$  which yields

$$T = \frac{1}{2\pi e^{-\beta} L} = \frac{1}{2\pi R}. \quad (3.19)$$

Hence we have again reproduced precisely the temperature emerging in our general considerations of CFT's in section 2.

Next we turn to the calculation of the horizon entropy, which we reduced to eq. (3.4) in the introductory remarks above. The expression there is simply proportional to the area of the horizon, which has the geometry  $H^{d-1}$ . As already noted, the total horizon

entropy diverges but this is precisely the expected result from section 2.2. Both the boundary and bulk description yield a uniform constant for the entropy density and so when integrated over the entire  $H^{d-1}$  geometry, it produces a divergent total entropy. As discussed in section 2.2, equality of this thermal entropy and the entanglement entropy requires a certain relation (2.33) between the long distance cut-off introduced to regulate the thermal entropy in  $R \times H^{d-1}$  and the short distance cut-off required to regulate the entanglement entropy across the sphere in Minkowski space. We now show that the same relation naturally appears in the holographic framework. In the Poincaré coordinates (3.9), a short distance cut-off  $\delta$  in the CFT is implemented by introducing a minimal radial coordinate  $z_{min}$  which cuts off the asymptotic region of the AdS geometry. The standard holographic dictionary for these two cut-offs is simply  $z_{min} = \delta$ . On the horizon (in particular, on the bifurcation surface), eq. (3.17) gave the relation  $\cosh u = e^{-\beta} L/z$  and so we find the maximum radius to regulate the calculation of the horizon entropy is naturally given by

$$\cosh u_{max} = \frac{e^{-\beta} L}{z_{min}} = \frac{R}{\delta}. \quad (3.20)$$

This relation can also be re-written as

$$\exp(-u_{max}) = \frac{R}{\delta} - \sqrt{\frac{R^2}{\delta^2} - 1} \simeq \frac{\delta}{2R}. \quad (3.21)$$

Hence, to leading order, we find agreement with the CFT expression in eq. (2.33).

We are now prepared to evaluate the horizon entropy (3.4). To facilitate a comparison with the results of [13, 14], we first make a change in the radial coordinate on the hyperbolic plane to  $x = \sinh u$ . Hence the IR regulator (3.20) becomes

$$\begin{aligned} x_{max} = \sinh u_{max} &= \sqrt{\frac{R^2}{\delta^2} - 1} \\ &= \frac{R}{\delta} \left( 1 + \frac{1}{2} \frac{\delta^2}{R^2} + \dots \right). \end{aligned} \quad (3.22)$$

Given the bulk metric (3.11), the horizon entropy (3.4) becomes

$$S = \frac{2\Gamma(d/2)}{\pi^{d/2-1}} a_d^* \Omega_{d-2} \int_0^{x_{max}} \frac{x^{d-2} dx}{\sqrt{1+x^2}}, \quad (3.23)$$

where  $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$  is the area of a unit  $(d-2)$ -sphere. This result precisely matches eq. (4.5) in [14] (up to the replacement  $x \rightarrow \rho$ ).

Hence as in [14], we observe that the leading contribution arising from eq. (3.23) can be written as

$$S \simeq \frac{2\pi}{\pi^{d/2}} \frac{\Gamma(d/2)}{d-2} a_d^* \frac{\mathcal{A}_{d-2}}{\delta^{d-2}} + \dots, \quad (3.24)$$



where  $\mathcal{A}_{d-2} = \Omega_{d-2} R^{d-2}$  is the ‘area’ of the entangling surface, *i.e.*, an  $S^{d-2}$  of radius  $R$ . Hence this leading divergence takes precisely the form expected for the ‘area law’ contribution to the entanglement entropy in a  $d$ -dimensional CFT [5, 10]. Further we note that the hyperbolic geometry of the horizon was essential to ensure the leading power was  $1/\delta^{d-2}$  here despite the area integral being  $(d-1)$ -dimensional in eq. (3.23). This divergent contribution to the entanglement entropy is not universal, *e.g.*, see [5, 10]. However, a universal contribution can be extracted from the subleading terms. The form of the universal contribution to the entanglement entropy depends on whether  $d$  is odd or even. For even  $d$ , the universal term is logarithmic in the cut-off while for odd  $d$ , it is simply a constant term [5, 10]. In the present case, expanding the entanglement entropy (3.23) in powers of  $R/\delta$ , we find the following universal contributions:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1} 4 a_d^* \log(2R/\delta) & \text{for even } d, \\ (-)^{\frac{d-1}{2}} 2\pi a_d^* & \text{for odd } d. \end{cases} \quad (3.25)$$

Further, recall that for even  $d$ ,  $a_d^* = A$ , the coefficient of the  $A$ -type trace anomaly in the boundary CFT.

### 3.2 Entanglement entropy in a cylindrical background

We now apply the results of our general CFT story discussion in section 2.3 to a holographic calculation of entanglement entropy in an  $R \times S^{d-1}$  background. In particular, we choose the entangling surface to be  $(d-2)$ -dimensional sphere of a fixed angular radius  $\theta_0$ . We begin, as before, with the description of  $\text{AdS}_{d+1}$  as a hyperboloid (3.5) embedded in  $R^{2,d}$ . In this case, we relate the embedding coordinates to the standard global coordinates on the AdS geometry

$$\begin{aligned} y_{-1} &= \tilde{\varrho} \cos(t/L), & y_0 &= \tilde{\varrho} \sin(t/L), & y_d &= \varrho \cos \theta, \\ y_1 &= \varrho \sin \theta \cos \phi_1, & y_2 &= \varrho \sin \theta \sin \phi_1 \cos \phi_2, & \dots & y_{d-1} = \varrho \sin \theta \sin \phi_1 \sin \phi_2 \dots \sin \phi_{d-2}. \end{aligned} \quad (3.26)$$

In this case, the constraint (3.5) yields  $\tilde{\varrho}^2 = \varrho^2 + L^2$  and the induced metric becomes

$$ds^2 = \frac{d\varrho^2}{\frac{\varrho^2}{L^2} + 1} - \left( \frac{\varrho^2}{L^2} + 1 \right) dt^2 + \varrho^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2). \quad (3.27)$$

If as usual, we consider the asymptotic limit  $\varrho \rightarrow \infty$  and remove a factor of  $\varrho^2/L^2$ , the boundary metric reduces to

$$ds_{\text{CFT}}^2 = -dt^2 + L^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2). \quad (3.28)$$

That is, we are studying the boundary CFT in the background geometry  $R \times S^{d-1}$ , where the radius of curvature of the sphere is  $L$ .



Now we wish to connect this coordinate choice to that giving the hyperbolic foliation in eqs. (3.15) and (3.16). For simplicity of the presentation, we again focus on the bifurcation surface in the corresponding metric (3.11). We approach the latter by taking  $\rho \rightarrow L$  with  $\tilde{\tau}$  fixed, which yields  $y_0 = 0$  and in turn we find  $t = 0$  in eq. (3.26). Then using eq. (3.16), we find that

$$\frac{y_d}{y_{-1}} = \frac{\varrho}{\sqrt{\varrho^2 + L^2}} \cos \theta = \tanh \beta. \quad (3.29)$$

Hence in the asymptotic limit  $\varrho \rightarrow \infty$ , we see that the bifurcation surface reaches an angular size on the  $S^{d-1}$  with

$$\cos \theta_0 = \tanh \beta \quad \text{or} \quad \sin \theta_0 = \frac{1}{\cosh \beta}. \quad (3.30)$$

This sphere is then the entangling surface in our calculation of entanglement entropy in the boundary CFT. We can further construct the full relation between the two coordinate systems and the result is identical to that given in eq. (2.35) with the substitutions:  $R = L$  and  $\tau = \tilde{\tau}$ .<sup>8</sup>

The entropy is again given by interpreting the hyperbolic foliation (3.11) as a topological black hole and evaluating the horizon entropy, as in eq. (3.4). However, we must first determine the appropriate IR regulator to introduce in the integral over the horizon. Given the global coordinates (3.27), the standard AdS/CFT dictionary introduces a short distance cut-off in the boundary CFT as a maximum radius:  $\varrho_{max} = L^2/\delta$ . Using the two expressions for  $y_{-1}$  in the two coordinate systems, we find that the bifurcation surface intersects this regulator surface at

$$\cosh u_{max} = \frac{L \sin \theta_0}{\delta} \left( 1 + \frac{\delta^2}{L^2} \right)^{1/2}. \quad (3.31)$$

We can relate this result to the CFT discussion in section 2.3 as follows. There a small angle  $\delta\theta$  was chosen to regulate the calculation of the entanglement entropy. This small angle defines a small proper distance on the  $S^{d-1}$  and so we relate this angular regulator to the short distance regulator above with  $\delta = L\delta\theta$ . Then a bit of algebra allows us to re-write eq. (3.31) as

$$\exp(-u_{max}) \simeq \frac{\delta\theta}{2 \sin \theta_0} + \dots. \quad (3.32)$$

Hence the holographic framework reproduces (to leading order) the relation (2.39) between the UV and IR regulators determined purely in our general discussion of CFT's.

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<sup>8</sup>Again, we originally derived the transformation (2.35) with this approach.

Our goal is to evaluate the holographic entanglement entropy across the entangling sphere at  $\theta = \theta_0$  which is again found by calculating the horizon entropy of the topological black hole (3.11). Hence we return to eq. (3.4) which we write as

$$S = \frac{2\Gamma(d/2)}{\pi^{d/2-1}} a_d^* \Omega_{d-2} \int_0^{x_{max}} \frac{x^{d-2} dx}{\sqrt{1+x^2}}, \quad (3.33)$$

where, as above, we have chosen  $x = \sinh u$  as an alternate radial coordinate on the hyperbolic plane  $H^{d-1}$ . With eq. (3.31), the regulator radius then becomes

$$x_{max} = \sinh u_{max} = \frac{L \sin \theta_0}{\delta} \left( 1 - \frac{\delta^2}{L^2} \cot^2 \theta_0 \right)^{1/2}. \quad (3.34)$$

With this result, we observe that the leading contribution arising from eq. (3.24) reproduces the expected ‘area law’ contribution to the entanglement entropy [5, 10]. The universal contributions, appearing in the subleading terms [5, 10], now take the form:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1} 4 a_d^* \log \left( \frac{2L}{\delta} \sin \theta_0 \right) & \text{for even } d, \\ (-)^{\frac{d-1}{2}} 2\pi a_d^* & \text{for odd } d. \end{cases} \quad (3.35)$$

Recall that here  $L$  denotes the radius of curvature of the  $S^{d-1}$ .

This derivation of the holographic entanglement entropy extends the calculations presented in [13, 14] which only examined the special case  $\theta_0 = \pi/2$ , *i.e.*, the entangling sphere was chosen to be the equator of the background  $S^{d-1}$  in the cylindrical background. Of course, setting  $\theta_0 = \pi/2$  above, we recover the results derived in [13, 14]. Note that this agreement is trivial for odd  $d$ , since the universal term is independent of  $\theta_0$  (as well as  $L$ ). Further we note that the above expressions are essentially the same (up to  $L \rightarrow R$ ) as those in eq. (3.25) where the background geometry was just flat Minkowski space.

## 4. A CFT calculation

In the previous section, we have found for a broad range of geometries that the universal contribution to the entanglement entropy for general holographic CFT’s is controlled by the central charge  $a_d^*$ . For even dimensional CFT’s, this charge coincides precisely with the coefficient of the  $A$ -type trace anomaly. Much of the motivation for this work came from [13, 14], where a similar result was found for the entanglement entropy between the two halves of the sphere in the background geometry  $R \times S^{d-1}$ . In fact, in [14], it was shown that their result actually applied for any CFT in even  $d$ , without reference to holography. Here we would like to show that our present results can also be extended to general CFT’s.

## 4.1 Mapping to de Sitter space

In order to calculate the entanglement entropy of the sphere for a CFT, we find it convenient to use a mapping to (the static patch of) de Sitter space, instead of the mapping to  $R \times H^{d-1}$ . We start again with the flat space metric for  $d$ -dimensional Minkowski space in polar coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2. \quad (4.1)$$

Now with the coordinate transformation

$$\begin{aligned} t &= R \frac{\cos \theta \sinh(\tau/R)}{1 + \cos \theta \cosh(\tau/R)}, \\ r &= R \frac{\sin \theta}{1 + \cos \theta \cosh(\tau/R)}, \end{aligned} \quad (4.2)$$

we can readily verify the flat space metric (4.1) becomes

$$\begin{aligned} ds^2 &= \Omega^2 \left[ -\cos^2 \theta d\tau^2 + R^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2) \right] \\ \text{where} \quad \Omega &= (1 + \cos \theta \cosh(\tau/R))^{-1}. \end{aligned} \quad (4.3)$$

After eliminating the conformal factor  $\Omega^2$ , the remaining metric corresponds to the static patch of  $d$ -dimensional de Sitter space with curvature scale  $R$ . The latter identification may be clearer if we transform to  $\hat{r} = R \sin \theta$ , which puts the above metric in the form

$$ds^2 = - \left( 1 - \frac{\hat{r}^2}{R^2} \right) d\tau^2 + \frac{d\hat{r}^2}{1 - \frac{\hat{r}^2}{R^2}} + \hat{r}^2 d\Omega_{d-2}^2. \quad (4.4)$$

With eq. (4.2), we observe that

$$\begin{aligned} \tau \rightarrow \pm\infty &: (t, r) \rightarrow (\pm R, 0) \\ \theta \rightarrow \frac{\pi}{2} &: (t, r) \rightarrow (0, R) \end{aligned} \quad (4.5)$$

Note that  $\theta = \pi/2$  corresponds to the cosmological horizon at the boundary of the static patch. In any event, with eq. (4.5), we see that the new coordinates cover precisely the causal development  $\mathcal{D}$  of the ball  $r \leq R$  on the surface  $t = 0$ .

Recall that inside  $\mathcal{D}$ , the modular transformations act geometrically along the flow in eq. (2.13). This transformation corresponds through the conformal mapping (4.2) to the time translation  $\tau \rightarrow \tau + 2\pi R s$  in de Sitter space — just as happened with the mapping to  $R \times H^{d-1}$ . Therefore the modular transformations act geometrically as time translations in the static patch and the state in the de Sitter geometry is thermal at temperature  $T = 1/(2\pi R)$  with respect to the Hamiltonian  $H_\tau$  generating  $\tau$  translations, *i.e.*, the density matrix is given by  $\rho \sim \exp[-2\pi R H_\tau]$ . Again, this result generalizes observations made in ref. [16], which examined conformal mappings of free conformal field theories in  $d = 4$ .

## 4.2 Thermodynamic entropy

As in section 2, the entanglement entropy for the sphere of radius  $R$  in flat space is equivalent to the thermodynamic entropy of the thermal state in de Sitter space. We are going to use standard thermodynamics in order to compute this entropy. Normalizing the thermal density matrix  $\rho = e^{-\beta H_\tau} / \text{tr}(e^{-\beta H_\tau})$ , we calculate the von Neumann entropy

$$\begin{aligned} S &= -\text{tr}(\rho \log \rho) = \beta \text{tr}(\rho H_\tau) + \log \text{tr}(e^{-\beta H_\tau}) \\ &= \beta E - W \end{aligned} \quad (4.6)$$

where  $W = -\log Z$  denotes the ‘free energy’ of the partition function  $Z = \text{tr}(\exp[-2\pi R H_\tau])$ .

The energy term in (4.6) is, of course, the expectation value of the operator which generates  $\tau$  translations. Since the latter translations correspond to a Killing symmetry of the static patch (4.3),  $E$  is just the Killing energy which can be expressed as

$$E = \int_V d^{d-1}x \sqrt{h} \langle T_{\mu\nu} \rangle \xi^\mu n^\nu = - \int_V d^{d-1}x \sqrt{-g} \langle T^\tau{}_\tau \rangle, \quad (4.7)$$

where the integral runs over  $V$ , a constant  $\tau$  slice out to  $\theta = \pi/2$ . Further  $n^\mu \partial_\mu \equiv \sqrt{|g^{\tau\tau}|} \partial_\tau$  is the unit vector normal to  $V$  and  $\xi^\mu \partial_\mu \equiv \partial_\tau$  is the time translation Killing vector.

Recall for the present investigation, the quantum field theory can be any general CFT. Then the state, coming from a conformal transformation of the Minkowski vacuum, is invariant under de Sitter symmetry group [36, 37] and we have

$$\langle T^\mu{}_\nu \rangle = \kappa \delta^\mu{}_\nu, \quad (4.8)$$

where  $\kappa$  is some constant. Hence the expectation value of the stress tensor is completely determined by the conformal anomaly (1.2). Note that in eq. (1.2), all of the the Weyl invariants vanish in de Sitter space, *i.e.*,  $I_n = 0$ , while the Euler density  $E_d$  yields a constant depending on the de Sitter radius  $R$ . Hence we can fix the constant  $\kappa$  in eq. (4.8) as

$$\langle T^\mu{}_\nu \rangle = -2(-)^{d/2} A \frac{E_d}{d} \delta^\mu{}_\nu, \quad (4.9)$$

for  $d$  even. The energy (4.7) then becomes

$$E = 2(-)^{d/2} A \frac{E_d}{d} R^{d-1} \Omega_{d-2} \int_0^{\pi/2} d\theta \cos \theta \sin^{d-2} \theta. \quad (4.10)$$

Hence the energy is finite. However, we are only interested in the universal coefficient of the logarithmic term in the entropy and hence, given this result, we may discard the

energy contribution in eq. (4.6). Note that for  $d$  odd, the trace anomaly vanishes and so we would find  $E = 0$ .

It remains to compute the contribution  $W = -\log \text{tr}(\exp[-2\pi R H_\tau])$  in eq. (4.6). This can be done as usual passing to imaginary time  $\tau_E$  and compactifying the Euclidean time with a period  $\beta = 2\pi R$ . The metric becomes

$$ds^2 = \cos^2\theta d\tau^2 + R^2 (d\theta^2 + \sin^2\theta d\Omega_{d-2}^2) . \quad (4.11)$$

This Euclidean manifold is precisely a  $d$ -dimensional sphere with radius of curvature  $R$ .<sup>9</sup> Note that the periodicity  $\Delta\tau = 2\pi R$  is precisely that required to avoid a conical singularity at  $\theta = \pi/2$ . Thus, one ends up with the Euclidean path integral on  $S^d$ .

Recall that we only need to determine the coefficient of the logarithmic term in the entropy for even  $d$ . Now the free energy has a general expansion

$$W = -\log Z = (\text{non-universal terms}) + a_{d+1} \log \delta + (\text{finite terms}) , \quad (4.12)$$

where  $\delta$  is our short distance cut-off and the non-universal terms diverge as inverse powers of  $\delta$ . The coefficient  $a_{d+1}$  for a conformal field theory is determined by the integrated conformal anomaly [39] — for free fields, it is one of the coefficients in the heat kernel expansion. In order to see this, consider an infinitesimal rescaling of the metric  $g^{\mu\nu} \rightarrow (1 - 2\delta\lambda)g^{\mu\nu}$ . Since<sup>10</sup>

$$\frac{2}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle + (\text{divergent terms}) , \quad (4.13)$$

in terms of the renormalized stress tensor  $\langle T_{\mu\nu} \rangle$ , we have

$$\frac{\delta W}{\delta \lambda} = - \int d^d x \sqrt{g} \langle T^\mu{}_\mu \rangle + (\text{divergent terms}) , \quad (4.14)$$

which is the integrated trace anomaly. On the other hand, due to the conformal invariance of the action, scaling the metric as above must give the same result as keeping the metric constant but shifting the UV regulator:  $\delta \rightarrow (1 - \delta\lambda)\delta$ . Combining these expressions with eq. (4.12), one finds

$$a_{d+1} = \int d^d x \sqrt{g} \langle T^\mu{}_\mu \rangle . \quad (4.15)$$

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<sup>9</sup>The fact that this metric (4.11) corresponds to the sphere may be more evident after the coordinate transformation [38]:  $\sin \theta = \sin \theta_1 \sin \theta_2$  and  $\tan(\tau/R) = \cos \theta_2 \tan \theta_1$ , which transforms the metric to

$$ds^2 = R^2 (d\theta_1^2 + \sin^2 \theta_1^2 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\Omega_{d-2}^2) .$$

<sup>10</sup>Recall we have made the transition to a Euclidean signature here.

Hence we are left to substitute eq. (1.2) for the trace anomaly and integrate over the  $S^d$ . Here we also need to observe that since the sphere is conformally flat, all of the Weyl invariants  $I_n$  vanish for the sphere, while our convention in eq. (1.2) is that the integral of the Euler density on  $S^d$  yields 2. Hence for any CFT in even dimensions, the universal contribution to the entanglement entropy becomes

$$S_{univ} = (-1)^{\frac{d}{2}-1} 4 A \log(R/\delta), \quad (4.16)$$

which is essentially the same as our holographic result in eq. (3.25). In particular, we see that the coefficient of the universal term in the entanglement entropy is proportional to the central charge  $A$ . Note that this result (4.16) and eq. (3.25) do not quite agree on the argument of the logarithm. In eq. (4.16),  $R$  was simply inserted as the only available scale in the problem whereas the result in eq. (3.25) emerged from a detailed evaluation of the entropy. Hence the mismatch is no surprise. However, the difference between the two expressions can be simply regarded as a non-universal constant term.

Although we do not present the details here, it is straightforward to extend our analysis above to the entanglement entropy across a sphere for CFT's in a cylindrical background  $R \times S^{d-1}$ . The result for the universal contribution for even  $d$  again matches our holographic result (3.35). In particular, the coefficient is controlled by the coefficient of the  $A$ -type trace anomaly and in fact, it is identical to that just above in eq. (4.16). As above, the present approach would not naturally reveal precisely the same scale over  $\delta$  in the argument of the logarithm, as we found in eq. (3.35).

In the odd dimensional case, we have seen  $E = 0$  and so eq. (4.6) reduces to  $S = \log Z$ . That is, the entanglement entropy is simply minus the free energy on a sphere. This case for odd dimensions has recently been examined in [22] for the particular case of a free scalar field and the results there are in agreement with this identification.

### 4.3 Hyperbolic mapping

It is interesting to go through a similar CFT analysis for our mapping (2.25) from flat space to  $R \times H^{d-1}$ . As discussed in section 2.2, in the latter hyperbolic space, we have a thermal state with  $T = 1/(2\pi R)$ . Now, however, this space is not maximally symmetric and so by symmetry, the stress tensor is only restricted to have a form

$$T^\mu{}_\nu = \text{diag}(-\mathcal{E}, p, \dots, p), \quad (4.17)$$

with  $\mathcal{E}$  and  $p$  constants. The background geometry is again conformally flat and so the Weyl invariants  $I_n$  vanish. Further, the background is the direct product of two lower dimensional geometries which dictates that the Euler density is also zero. Hence the

trace anomaly (1.2) vanishes in this particular background and in eq. (4.17), we must have

$$\mathcal{E} = (d - 1) p. \quad (4.18)$$

To proceed further we must focus on  $d = 4$  since in general, we do not know the value of the energy density. However, for  $d = 4$ , we have the Bunch-Davies-Brown-Cassidy formula [40] which relates the stress tensor for a given state in flat space to that in another space obtained by a conformal mapping from flat space. With this expression, we obtain

$$\mathcal{E} = \frac{3a + c}{8\pi^2 R^4}. \quad (4.19)$$

Here, we have adopted the standard notation for the central charges in four-dimensional CFT's, *i.e.*, comparing to eq. (1.2),  $a = A$  and  $c = 16\pi^2 B_1$  [13].<sup>11</sup>

Now, in contrast to the discussion in the previous section, the energy contribution in eq. (4.6) is divergent due to the infinite volume of the  $H^{d-1}$ . Using eq. (4.19) for  $d = 4$ , we find a contribution to the logarithmic term in the entropy

$$\begin{aligned} 2\pi R E &= 8\pi^2 R^4 \mathcal{E} \int_0^{u_{max}} du \sinh^2(u) \\ &= (\text{non-universal terms}) + \frac{1}{2} (3a + c) \log \delta + (\text{finite terms}). \end{aligned} \quad (4.20)$$

The final result above relies on choosing the maximum radius  $u_{max}$  as in eq. (2.33).

The second contribution to the logarithmic term in the entropy (4.6) comes from the free energy  $W$  which should again be determined by the conformal anomaly, as in eqs. (4.12) and (4.15). However, as noted above,  $\langle T^\mu_\mu \rangle = 0$  in the present case and so we have a zero ‘bulk’ contribution from the anomaly. However, implicitly, the manifold comes with a boundary, *i.e.*,  $u = u_{max}$  in the cut-off manifold. Hence the conformal anomaly should pick up boundary contributions there but unfortunately, at present, the expression for these terms is unknown. One could take the results for the entropy (4.16) and the energy (4.20) in  $d = 4$  to determine this boundary contribution as

$$W = (\text{non-universal terms}) + \frac{1}{2} (-5a + c) \log \delta. \quad (4.21)$$

In particular, the term proportional to  $c$  in eq. (4.20) must cancel with a boundary term. It is intriguing that this approach may imply that there are further restrictions for the boundary conditions that should be used to define the cut-off at  $u = u_{max}$ .

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<sup>11</sup>Our conventions are such that these coefficients are normalized to  $a = 1/360$  and  $c = 1/120$  for a free conformally coupled massless (real) scalar.

## 5. Discussion

To summarize our results, we have produced a derivation of holographic entanglement entropy for certain geometries, namely, with spherical entangling surfaces. The derivation started by finding an appropriate calculation of the entanglement entropy in the boundary CFT in section 2. Here, we have avoided the usual approach of using the replica trick [2, 12]. Rather we used conformal transformations to relate the entanglement entropy across a spherical entangling surface to the thermal entropy in a new background geometry,  $R \times H^{d-1}$ . While this construction applies for any CFT, it is not particularly useful in general as it simply relates two difficult problems to one another. However, in the case of a holographic CFT, the AdS/CFT correspondence translates the second problem to the question of determining the horizon entropy of a topological black hole, as described in section 3. The latter is a straightforward calculation using Wald’s entropy formula (3.2). Hence we have derived an expression for the holographic entanglement entropy (3.4), which applies for any bulk gravitational theory, albeit for the specialized case of a spherical entangling surface. We have explicitly considered the entanglement entropy for flat space, in section 3.1, or for a cylindrical background, in section 3.2. This discussion would also straightforwardly extend to a spherical entangling surface in a background  $R \times H^{d-1}$ . The key difference, however, would be that we would not start with the vacuum state in this background, rather we take the thermal state that is equivalent to the vacuum in  $R^d$  by the conformal mappings introduced in section 2.

The present discussion extends the derivation presented in [14], where the analysis focused on a special case of the geometries considered here. There the background geometry was chosen to be  $R \times S^{d-1}$  and the entangling surface was placed on the equator of the  $S^{d-1}$ . We might comment, however, that the enhanced symmetry of the latter geometry allowed for an alternate derivation which was based on the replica trick.

In section 2, both the entanglement and thermal entropy were divergent and their equality was only guaranteed by imposing a relation between the short-distance cut-off required to regulate the entanglement entropy and the long-distance cut-off used to regulate the thermal entropy, as in eqs. (2.33) and (2.39). Hence the conformal mapping introduced an interesting UV/IR relation between the two states of the CFT, which is reminiscent of the UV/IR connection found in the AdS/CFT correspondence. The holographic dictionary also naturally reproduced these relations in section 3. In this case, the horizon of the topological black hole extended out to the AdS boundary and the desired relations were determined examining the intersection of the horizon with the UV regulator surface associated with the original background geometry. The



intersection of these two surfaces was also the key feature which distinguished different horizons connected to boundary spheres with different sizes. Otherwise the horizons are ‘identical’, in that, these surfaces can all be mapped into one another by an isometry of the  $\text{AdS}_{d+1}$  geometry.

Examining the results in eqs. (3.25) and (3.35), we observe that the universal contribution to the entanglement entropy is proportional to the central charge,  $a_d^*$ , which characterizes the boundary CFT. This charge was introduced in [13, 14] where it was observed that this charge satisfies a holographic c-theorem<sup>12</sup> — a result which was conjectured to extend to general field theories. Following the considerations of the holographic principle in [45], it was also argued that  $a_d^*$  gave a measure of the number of degrees of freedom in the boundary field theory [14]. Given the present results, we can identify this charge using entanglement entropy for a broader class of geometries, in particular, if we wish to examine the c-theorem noted above outside of a holographic framework. Specifically, with eq. (3.25),  $a_d^*$  can always be identified using a spherical entangling surface in flat space.

For the case of an even dimensional boundary theory (*i.e.*,  $d$  even), the central charge is precisely that appearing in the  $A$ -type trace anomaly (1.2). In this case, the universal term in the entanglement entropy is proportional to a logarithm of the cut-off scale  $\delta$ . Of course, the latter is balanced by another scale to make the argument of the logarithm dimensionless. In the flat Minkowski background, this scale is naturally set by the size of the entangling sphere, as in eq. (3.25), since this is the only scale in the construction. In our result (3.35) for the Einstein universe background, the other scale is  $L \sin \theta_0$  (up to a factor of 2, which also appears in the flat space calculation). This scale can be readily identified as the radius of curvature of the entangling sphere. Hence, in this sense, the same scale appears in the result for both of our calculations.

We also note that our result (3.35) is a simple generalization of the standard result for the entanglement entropy of a two-dimensional CFT. Given a sub-system of length  $\ell$  in a full system of size  $C$  (with periodic boundary conditions), the entanglement entropy is given by [2, 46]

$$S = \frac{c}{3} \log \left( \frac{C}{\pi \delta} \sin \frac{\pi \ell}{C} \right), \quad (5.1)$$

where  $c$  is the central charge of the  $d = 2$  CFT.<sup>13</sup> Of course, for two dimensions, the cylindrical background considered in section 3.2 reduces to  $R \times S^1$ . Further our result (3.35) precisely matches the result given above using the relations:  $a_d^* = A = c/12$ ,  $L = C/2\pi$  and  $\theta_0 = \pi \ell / C$ , which are applicable for  $d = 2$ . Our result in eq. (3.35) provides a natural extension of the above expression for  $d = 2$  to higher (even) dimensions.

<sup>12</sup>See also recent related results in [41, 42, 43, 44].

<sup>13</sup>In general, one might also expect a non-universal constant term to appear on the right-hand side.

Of course, in section 4, we were able to show that the universal contribution to the entanglement entropy takes the same form as our holographic result (3.25) for any CFT in any even number of dimensions. A similar result was proven in [14] where the entangling surface was the equator of the  $S^{d-1}$  in the static Einstein universe background. While we did not present the details, it would be straightforward to extend the arguments of section 4 to cover this case or a spherical entangling surface of any angular size. Hence, for even  $d$  (without any reference to holography), we have established that the universal contribution to the entanglement entropy across a spherical entangling sphere is proportional to  $A$ .

The calculation in section 4 begins with a mapping of the causal development  $\mathcal{D}$  to the static patch in de Sitter space. This approach is similar to that in [19]. However, in the latter, the thermodynamical entropy is written as

$$S = \int_0^{T=\frac{1}{2\pi R}} \frac{dE(T, V)}{T} \Big|_{V=\text{const.}}. \quad (5.2)$$

The main difference<sup>14</sup> with our approach is that in eq. (5.2), one needs to know the energy ‘off shell’, *i.e.*, away from the point  $T = 1/(2\pi R)$ . This thermal energy is known only for the case of free fields and so the results in [19] are only determined for free fields. In contrast, our ‘on shell’ approach only makes reference to the energy and free energy at  $T = 1/(2\pi R)$  which naturally emerges from the conformal transformation of the Minkowski vacuum. This essential difference allowed us to derive a general result for any CFT in any even number of dimensions. However, we might add that it seems a knowledge of the whole function  $E(T, V)$  would certainly be necessary for computing the Renyi entropies  $S_n = (1 - n)^{-1} \log(\text{tr} \rho^n)$  for general  $n$ .

The general connection of the universal terms in entanglement entropy and the trace anomaly was first noted in [46] for  $d = 2$  and extended to higher dimensions in [5, 17]. In particular, [17] established that the entanglement entropy for a spherical entangling surface would be proportional to  $A$  in four dimensions. However, we must comment that the arguments presented in [5, 17] are only completely justified when there is a rotational symmetry in the transverse space around the entangling surface. For configurations without this symmetry, additional correction terms must be added to the entanglement entropy, however, they still seem to have the same general character, *i.e.*, they are geometric expressions evaluated on the entangling surface with coefficients linear in the central charges [11, 17, 47].

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<sup>14</sup>This approach must also assume that the entropy at zero temperature vanishes or at least the relevant logarithmic contribution vanishes. This assumption may be related to the mismatch in [19], where the coefficient to the logarithmic term is not given by the  $A$ -type anomaly for a vector field.

We note that the coefficient of the universal contribution is identical for any sphere of any size in flat space or in a cylindrical background. Given that all of these geometries are related by conformal mappings, this reflects the fact that this coefficient is conformally invariant, as noted in [20], and clarified in the present paper. Another case then, which also refers to a spherical entangling surface was recently discussed in [21]. This case is the near-horizon geometry of an extreme black hole, which has a geometry  $H^2 \times S^{d-2}$  and by explicit calculation for a free conformally coupled scalar in any even dimension, it was shown that the coefficient of the log term was controlled by  $A$ . The event horizon in this geometry can be conformally mapped to a sphere in flat space. Hence one expects the coefficient for the logarithmic correction to the black hole entropy is also controlled by the  $A$ -type trace anomaly for even  $d$ . In fact, given the conformal invariance of this coefficient, the discussion in [14] would be sufficient to indicate that  $A$  also appears as the coefficient of the universal contribution for any CFT in geometries considered in section 4.

Our calculations of holographic entanglement entropy also yielded results for odd  $d$  in eqs. (3.25) and (3.35). In this case, following [5], we have identified the universal contribution as the constant term appearing in the expansion in powers of the cut-off. Hence the result is completely independent of the size of the entangling sphere or the background geometry. The universality of this constant contribution to the entanglement entropy is established for a variety of three-dimensional systems [1, 3]. However, we should note that one may worry that in general the precise value of this constant will depend on the details of the regulator — as discussed in [14].<sup>15</sup> We expect that these issues can be circumvented by considering an appropriate construction with mutual information — *e.g.*, see [48, 49]. When calculated with two separate regions, the latter is free of divergences and any regulator ambiguities.

Recall our result in section 4 for the case of odd dimensional CFT's. Namely, the entanglement entropy for a spherical entangling surface  $S^{d-2}$  is precisely minus the free energy of the CFT on a sphere  $S^d$ , *i.e.*, we have

$$S = \log Z \quad \text{for odd } d. \quad (5.3)$$

This result can be related to a calculation of entanglement entropy described in [14]. There, the initial problem was to determine the entanglement entropy of a  $d$ -dimensional CFT on  $R \times S^{d-1}$  when the entangling surface  $\Sigma$  was chosen to be the equator of the sphere. The approach taken was to apply the geometric approach to the replica trick [12], where one evaluates the partition function on the background geometry with an infinitesimal conical defect at  $\Sigma$ . However, this procedure is only well-defined if there is

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<sup>15</sup>RCM thanks M. Smolkin, A. Schwimmer and S. Theisen for discussions on this point.

a rotational symmetry about this surface and so in [14], the  $R \times S^{d-1}$  background was mapped to  $S^d$  using a conformal transformation. The expression for the entanglement entropy then becomes

$$S = \lim_{\epsilon \rightarrow 0} \left( \frac{\partial}{\partial \epsilon} + 1 \right) \log Z_{1-\epsilon}. \quad (5.4)$$

where the partition function is evaluated on a ‘ $(1 - \epsilon)$ -fold cover’ of the  $d$ -dimensional sphere, with an infinitesimal conical defect  $\Delta\theta = 2\pi(1 - \epsilon)$  at  $\Sigma$ . In [14], this construction was applied to determine the universal contribution to the entanglement entropy for even  $d$ . However, it applies just as well for the case of odd  $d$ . Hence comparing to eqs. (5.3) and (5.4), we see that the leading variation of the sphere partition function must vanish in the latter equation, *i.e.*,  $\partial_\epsilon Z|_{\epsilon=0} = 0$ . This vanishing of the variation of  $Z$  is analogous to the vanishing of the energy contribution in eq. (4.6) for odd  $d$ .

We should note that the expressions on both sides of eq. (5.3) are expected to diverge and so, as stressed in section 2, care must be taken in applying a consistent regulator to ensure the equality of the two quantities. As noted before, this equality was established for free conformal scalar fields in odd dimensions by [22]. There, in fact, the heat kernel regulator completely eliminated the divergences on both sides of eq. (5.3). With a general regulator, where divergent terms still appear, one still expects that  $S$  and  $\log Z$  will be equal order by order in the cut-off scale. In particular, the universal constant contribution to the entanglement entropy must match the constant term in free energy.

This last observation is of interest in connection to a recent discussion of  $\mathcal{N} = 2$  superconformal field theories in three dimensions in [50]. There the author provides evidence that, for these theories, the sphere partition function plays a very similar role to the central charge  $a$  in four-dimensional theories with  $\mathcal{N} = 1$  supersymmetry. In particular, as function of possible trial  $R$ -charges, (the finite part of)  $Z$  is extremized by the exact superconformal  $R$ -charge, in analogy to  $a$ -maximization in four dimensions [51]. Further, given the connection between  $a$ -maximization and the c-theorem for the corresponding field theories [52], one is naturally lead to speculate that  $Z$ -maximization may provide a framework to develop the analog of c-theorem for supersymmetric theories in three dimensions [50]. Now, the identity in eq. (5.3) connects this suggestion to the broader conjecture of [13, 14]. Motivated by holographic evidence, the authors there proposed that the central charge  $a_d^*$ , which appears in the universal contribution to the entanglement entropy, should evolve monotonically under RG flows. Focussing on three dimensions, eq. (5.3) would yield

$$\log Z|_{finite} = S_{univ} = -2\pi a_3^*. \quad (5.5)$$

Hence the holographic results of [13, 14] indicate that any  $d = 3$  supersymmetric gauge theories with a gravity dual will satisfy the desired c-theorem. Alternatively, if the field theoretic approach of [50] can be extended to establish a c-theorem as a consequence of  $Z$ -maximization, this would provide further evidence for the general c-theorem conjectured in [13, 14] for quantum field theories in any (odd) number of spacetime dimensions

Given our new results, it is interesting to consider some explicit applications, in particular, to consider higher curvature corrections in the holographic entanglement entropy in various string models. A well-known set of corrections appear at order (curvature)<sup>4</sup> in all superstring theories [53]. The supersymmetric completion of this term in type IIB string theory was used to explicitly construct all of the interactions involving the curvature and the Ramond-Ramond five-form [54]. These interactions are all naturally written in terms of the Weyl tensor and certain tensors constructed from the five-form. Now in a holographic setting, we are considering a supersymmetric reduction of the form  $\text{AdS}_5 \times \mathcal{M}_5$  and in such a background, both the Weyl tensor and the relevant tensors for the five-form vanish. Hence these interactions do not modify the background geometry nor do they contribute to the Wald entropy (3.2) of the topological black hole. That is, these particular higher order corrections to the superstring action leave the entanglement entropy unchanged. Note that implicitly we have extended the discussion to the full ten dimensions of the superstring theory and so the horizon has the geometry  $H^3 \times \mathcal{M}_5$ . We could first reduce the ten-dimensional theory to five dimensions, following [55], and present the argument entirely in the context of  $\text{AdS}_5$  as in the main text. Of course, the result is unchanged.

Further, this result, namely that the entanglement entropy was unchanged, should have been expected. These  $R^4$  interactions appear at order  $\alpha'^3$  with a tree-level and one-loop (*i.e.*,  $g_s^2$ ) contribution. From the perspective of the boundary theory then, these terms will introduce corrections of order  $1/\lambda^{3/2}$  and  $\lambda^{1/2}/N_c^2$  [56]. In particular then, these depend on the 't Hooft coupling  $\lambda$ . However, our analysis indicates that the universal contribution to the entanglement entropy should be proportion to the central charge  $A$  (which is commonly denoted  $a$  in four dimensions). Further in a superconformal gauge theory, it is known that the central charges are independent of the gauge coupling [57]. Hence, this universal contribution should not receive any corrections depending on the 't Hooft coupling, which is in accord with our gravity calculations. We might add that for  $N = 4$  super-Yang-Mills theory in the limit of zero coupling (*i.e.*, the free field limit), numerical calculations [58] of the entanglement entropy for a spherical entangling surface in flat space explicitly confirm that the results match the strong coupling result. Hence these calculations also confirm the same independence of the gauge coupling.

There are other interesting string theory models where curvature-squared terms arise in a holographic context [59]. These terms originate from the presence of D-branes in the construction of these backgrounds [60]. We do not present the details here but we comment that the presence of these terms does effect the two central charges,  $a$  and  $c$ , of the dual CFT. In particular, the difference  $c - a$  is controlled by the coefficient of this higher curvature interaction. However, we are again discussing these terms in the context where they appear in a controlled perturbative expansion in string theory and hence they can be modified by field redefinitions. In particular then, if this interaction is written as  $R_{abcd}R^{abcd}$ , it contributes by modifying both the background curvature of the  $\text{AdS}_5$  and the expression for the central charge  $A$ . Of course, this term also contributes to the Wald entropy and the modifications are consistent with are final result (3.25) and (3.35) where  $A$  appears in the coefficient of the universal contribution to the entanglement entropy. However, with field redefinitions, the interaction can also be written as  $C_{abcd}C^{abcd}$ , in which case, neither the background curvature nor the central charge  $a$  are modified. Further, this term will not contribute to the Wald entropy and so again the results are consistent with our expressions for the entanglement entropy.

One issue which our discussion highlights is the close connection between holographic entanglement entropy and black hole entropy. Indeed an eternal AdS black hole contains two asymptotically AdS regions and it has been argued that in this case the horizon entropy corresponds to the entanglement entropy between the CFT on one boundary and its thermofield double on the other boundary [61]. Our construction essentially uses this interpretation for the topological black hole which corresponds to a pure AdS spacetime. The key difference from the usual interpretation is that the two asymptotically AdS regions are complementary portions of the same AdS geometry. It would be interesting to understand if a similar interpretation is possible for topological black holes which are endowed with charge charge and/or rotation [62]. Another useful direction would be to investigate whether AdS space can be foliated in other ways to produce ‘topological black holes’ with different horizon geometries. These may then form the basis of a derivation of the holographic entanglement entropy for entangling surfaces with new geometries.

Of course, our derivation puts the standard proposal of [4, 5] on a firmer footing since we find agreement with eq. (1.1) when the bulk theory is just Einstein gravity. We might note that the present calculations do not seem to involve the extremization of some functional over a family of bulk surfaces. However, given our results, a natural guess might be that when the bulk gravity theory includes higher curvature terms, we should extend the definition of holographic entanglement entropy (1.1) to extremize the Wald entropy (3.2) evaluated on bulk surfaces homologous to the boundary region of interest. Unfortunately, one can easily show that this procedure does not produce the

correct entanglement entropy in general, however, interesting progress has still been made for certain classes of higher curvature theories [11, 41]. At the same time, we can add that for generic entangling surfaces, the bulk surface determining the holographic entanglement entropy will not play the role of the event horizon for some black hole [11, 41].

Again, our derivation of holographic entanglement entropy discards the replica trick and instead we are relying on invariance of the entanglement entropy of the boundary CFT under conformal mappings. Of course, it would be interesting to extend our approach to produce a derivation for more general geometries, *i.e.*, non-spherical entangling surfaces in different background spacetimes. A common feature of the conformal mappings in sections 3.1 and 3.2, which seems important, is that the entangling sphere is mapped to space-like infinity in  $R \times H^{d-1}$ . While similar transformations mapping the entangling surface to infinity are easily constructed for other geometries, *e.g.*,  $S^{d-2-n} \times R^n$ , the resulting background is typically time-dependent and it is not evident what the state of the CFT is in the new background. Hence if further progress is to be made with this approach, additional insights will be needed with respect to the most useful conformal mapping to apply for a given geometry. It may be instructive to translate the discussion of section 4 to a holographic derivation of the entanglement entropy of a spherical entangling surface. In any event, there remain many interesting questions to explore with regards to holographic entanglement entropy.

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